# MA exam: Algebra — 20 April 2019

Please do four problems, including one from each of the three sections. Give complete proofs — do more than simply quote a theorem. Please indicate clearly which four problems you want to have graded.

## Part I: Group theory

**1.** Let $G$ be a finite abelian group and let $z$ be the product of all of the elements in $G$.

- **(a)** Prove that $z^2 = 1$.
- **(b)** Give an example of $G \neq 1$ where $z = 1$.
- **(c)** Give an example of $G$ where $z \neq 1$.

**2.** In this problem $G$ is always a finite group.

- **(a)** Show that if the order of $G$ satisfies $|G| = p^n$ where $p$ is a prime integer and $n$ a positive integer, then the center $Z(G)$ is non-trivial.
- **(b)** Show that if the order of $G$ satisfies $|G| = pq$ where $p$ and $q$ are distinct primes, then $G$ cannot be a *simple* group (i.e., $G$ must contain a non-trivial normal subgroup).

## Part II: Ring theory

**3.** Let $R$ be a commutative ring with an identity element. Under addition $R$ is, of course, an abelian group. Suppose that each subgroup of this group is, in fact, an ideal of $R$. Show that the ring $R$ is isomorphic to the ring of integers $\mathbb{Z}$, or to the integers modulo $n$, for some integer $n$.

**4.** Recall that an ideal $P$ in a commutative ring $R$ is *prime* if, whenever $x, y \in R$ are such that $xy \in P$, then either $x \in P$ or $y \in P$ (or possibly both).

Let $\mathbb{Q}$ be the field of rational numbers, $\mathbb{Z}$ be the ring of integers, with $\mathbb{Q}[x]$ and $\mathbb{Z}[x]$ the corresponding polynomial rings.

- **(a)** Show that in $\mathbb{Q}[x]$ every prime ideal is a maximal ideal.
- **(b)** Exhibit a prime ideal in $\mathbb{Z}[x]$ that is *not* maximal.

## Part III: Linear algebra

**5.** In this problem, $A$ is an $m \times m$ matrix over the field of real numbers such that $A^n = I$ (the identity matrix) for some positive integer $n$.

- **(a)** Prove that $A^2 = I$ if such an $A$ is symmetric.
- **(b)** Give an example of such an $A$ where $A^2 \neq I$. Of course, by part (a), this $A$ won't be symmetric.

**6.** Let $A$ be the $4 \times 4$ real matrix

$$\begin{pmatrix} 0 & 4 & 0 & 4 \\ 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- **(a)** Find the characteristic polynomial of $A$ and the eigenvalues of $A$.
- **(b)** Find a basis for each eigenspace of $A$.
- **(c)** Find $J$, the Jordan canonical form of $A$.
- **(d)** Find an invertible matrix $P$ such that $AP = PJ$.