G. Martin & E. Shemyakova

Answer any 5 questions. If more that 5 questions are attempted, credit will be given for the best 5 answers.
You have two hours.

1. Prove or disprove that the set of real numbers $\mathbb{R}$ with the finite-complement topology is Hausdorff topological space.

2. Consider the set $X = \{a, b, c\}$ with the topology $\{\emptyset, \{a\}, \{a, b\}, X\}$, and the closed interval $[0, 1]$ with the topology induced from $\mathbb{R}$. Give an example of a non-constant continuous function $f : [0, 1] \to X$. Prove that your function is continuous.

3. Consider topological spaces X and Y and let Y be compact. Let $\pi_1 : X \times Y \to X$ be the projection map, $(x, y) \mapsto x$. Show that π_1 is a closed map.

4. A topological space X is a *regular* space if, given any closed set F and any point x that does not belong to F, there exist an open set U such that $x \in U$ and an open set V such that $F \subset V$ that are disjoint, $U \cap V = \emptyset$. Show that a compact Hausdorff space is regular.

5. A topological space is called *countably compact* if every countable open cover contains a finite subcover. A topological space X is called *sequentially compact* if every sequence of points in X has a convergent subsequence converging to a point in X.

   Prove that every sequentially compact space is countably compact.

6. Define the fundamental group of a topological space at a given point. Define the group operation, the identity element and the inverse.

   Prove that the identity and the inverse indeed satisfy the required properties. (You do not need to prove the associativity.)

7. Define what is meant by a covering map. Give examples of two (different from identity) covering maps: one for S^1, and one for $S^1 \times S^1$. Give formulas. Prove that they are coverings.