M.S. and M.A. Comprehensive Analysis Exam

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To get full credit you must show all your work.
This exam contains 6 real analysis and 6 complex variables questions.

Real Analysis

100% will be obtained for complete answers to four questions. Indicate clearly which four questions you wish to be graded.

1. Define $f(x) = \begin{cases} 0 & \text{if } x = 0, \\ x^3 \cos(\frac{1}{x}) & \text{if } x \neq 0. \end{cases}$

   Show that f' is continuous at 0 but f' is not differentiable at 0.

2. (a) Let f be a bounded function on $[a, b]$. Define the Riemann integral $\int_a^b f$.

   (b) Define $g(x) = \begin{cases} -1 & \text{if } x \text{ is irrational}, \\ x & \text{if } x \text{ is rational}. \end{cases}$

   Show that g is **not** Riemann integrable on $[0, 1]$.

3. (a) Define what it means that a function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

   (b) Show that $f(x) = \sin(x)$ is uniformly continuous on $\mathbb{R}$.

   (c) Show that $g(x) = \sin(x^2)$ is **not** uniformly continuous on $\mathbb{R}$.

4. Let (X, d) be a metric space and define $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)}$ for all $x, y \in X$. Show that ρ is a metric on X.

5. For $n = 1, 2, \ldots$ and $0 \leq x \leq 1$, define $f_n(x) = x^n$.

   (a) Fix $0 < a < 1$. Show that the sequence $\{f_n\}$ converges uniformly on $(0, a)$.

   (b) Is there a subsequence $\{f_{n_k}\}$ that converges uniformly on $(0, 1)$? Explain.

6. Let (X, d) be a metric space and $f : X \to \mathbb{R}$ be a continuous function. If X is compact, show that f is uniformly continuous.

Complex Analysis

100% will be obtained for complete answers to four questions. Indicate clearly which four questions you wish to be graded.

1. Let $f(z) = \dfrac{z-i}{z+i}$. Show that f maps the upper half-plane $\{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ into the unit disk $\{z \in \mathbb{C} : |z| < 1\}$.

2. Evaluate the following integral, where C is the positively oriented circle centered at $(2, 0)$ with radius 2:

$$\oint_C \frac{ze^{3z}}{(z^2-1)^2} dz.$$

3. Expand $f(z) = \dfrac{z}{(z+i)(z-3)}$ in a Laurent series valid for $\{z \in \mathbb{C} : 1 < |z| < 3\}$.

4. Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Assume that $u(x,y) \neq v(x,y)$ for all $(x,y) \in \mathbb{D}$. Find all such f for which the function

$$g(z) = [u(x,y)]^2 + i[v(x,y)]^2$$

   is analytic in $\mathbb{D}$.

5. Evaluate the integral $\displaystyle\int_0^\infty \frac{x^2}{(x^2+1)(x^2+9)}\, dx$.

6. Find all possible entire functions f with the property that $|f(z)| \leq 2|z| + 1$ for all $z \in \mathbb{C}$. Prove that you have found all such functions.