MS Comprehensive EXAM
DIFFERENTIAL EQUATIONS
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This exam has two parts, ordinary differential equations and partial differential equations. Choose four problems in each part

Part I: Ordinary Differential Equations

1. Find the solution of the initial value problem on $(0, \infty)$ Draw the graph of the solution as a parametric curve

$$x'(t) = y(t),\ y'(t) = -5x(t) - 2y(t),\ \ x(0) = 1,\ \ y'(0) = -2$$

2. Find three linearly independent solutions of the equation and show that these solutions are linearly independent.

$$y''' + 3y'' + 3y'(t) + y = 0$$

3. Prove that there are no continuous functions $p(t), q(t)$ on $\mathbb{R}$ so that $y_1(t) = e^t$ and $y_2(t) = t^2$ are both solutions for $y'' + p(t)y' + q(t)y = 0$.

4. Draw the phase portrait of the differential equation on the half plane $x \geq 0$ Find the smallest v_0 such that the solution $x(t)$ of the equation with the initial conditions $x(0) = 0, \dot{x}(0) = v_0$ satisfies $\lim_{t\to\infty} x(t) = \infty$

$$\ddot{x} = -\frac{24}{(6+x)^2}$$

5. Let $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

1 Find the eigenvalues and corresponding eigenvectors of A

2 Draw phase portrait of $X' = AX$ near the origin and determine the stability of the origin

3 Solve $X' = AX$ by computing e^{tA}

6. Compute eigenvalues and eigenfunctions of the following Sturm-Liouville problem.

$$y''(x) + \lambda y(x) = 0;\quad y(0) = y(2\pi),\ y'(0) = y'(2\pi)$$

7. Suppose $y(t)$ has continuous first derivative and satisfies $|y'(t)| \leq 2y(t)$ on [-1 1] and $y(0) = 0$ Show that $y(t)$ is identically equal to zero on [-1,1].

Part II: Partial Differential Equations

1. Find a solution of the initial value problem for the heat equation

$$\begin{cases} u_t(x,t) = 4u_{xx}(x,t) & \text{on } -\infty < x < \infty,\ t > 0, \\ u(x,0) = 3e^{-2x^2} + 2 & \text{for } -\infty < x < \infty \end{cases}$$

2. Find the general solution for $u_{tt} - 3u_{xt} + 2u_{xx} = 0$ on $\mathbb{R}^2$

3. Find all functions $u(x,y)$ satisfying $\Delta u + \lambda u = 0$, where λ is a number, inside the unit square $[0,\pi] \times [0,\pi]$ and vanishing on its boundary

4. Solve the initial-boundary value problem for the wave equation

$$\begin{cases} u_{tt}(x,t) = 9u_{xx}(x,t) & \text{on } 0 \le x \le \pi, t > 0, \\ u(0,t) = 0, \quad u(\pi,t) = 0 & \text{for } t > 0, \\ u(x,0) = 3\sin(x) - \frac{1}{4}\sin(3x) + 2\sin(5x), & \\ u_t(x,0) = 0 & \text{for } 0 \le x \le \pi \end{cases}$$

5. Show that the following boundary value problem has at most one solution

$$\begin{cases} u_t = u_{xx}, & 0 < x < 1, t > 0 \\ u(x,0) = f(x), & 0 \le x \le 1 \\ u(0,t) = g(t), & t \ge 0 \\ u(1,t) = h(t), & t \ge 0 \end{cases}$$

6. Let $u(x,y) = (x,y) = -x^2 + y^2 + y^3 - 3x^2y$ Find $v(x,y)$ satisfying the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

and the condition $v(0,0) = 0$

7. Let Ω be a bounded domain in $\mathbb{R}^n$ Suppose a C^2 function u satisfies $-\Delta u \le 0$ inside the domain and $u \le 0$ on its boundary Prove that $u(x) \le 0$ inside the domain