Department of Mathematics
University of Toledo

Master of Science Degree
Comprehensive Examination
Probability and Statistical Theory

May 18, 1994

Instructions:
Do any FIVE problems out of the six given.
Show all of your computations in your Blue Book.
Prove all of your assertions or quote appropriate theorems.
Books, notes, and calculators *may be used.*
This is a three hour test.

1. Let $X_1, \ldots, X_n$ be a random sample from a Poisson distribution with mean λ, where λ is unknown.

(a) Find the Fisher information about λ contained in $X_1, \ldots, X_n$.

(b) Let $K > 0$ be a known integer. Find the maximum likelihood estimator (MLE) of $\frac{\lambda^K e^{-\lambda}}{K!}$.

(c) Let $K > 0$ be a known integer. Find the uniformly minimum variance unbiased estimator (UMVUE) of $\frac{\lambda^K e^{-\lambda}}{K!}$.

For the following problem, you should employ the following table for the Binomial(n=4,p) distribution.

x	p=.2	p=.4	p=.5	p=.6	p=.8
0	0.4096	0.1296	0.0625	0.0256	0.0016
1	0.4096	0.3456	0.2500	0.1536	0.0256
2	0.1536	0.3456	0.3750	0.3456	0.1536
3	0.0256	0.1536	0.2500	0.3456	0.4096
4	0.0016	0.0256	0.0625	0.1296	0.4096

Also, a Bayes hypothesis test of $H_0{:}\theta\in A$ versus $H_1{:}\theta\notin A$ based on observations given by X at level of significance α is given by the following simple rule:

Reject H_0 in favor of H_1 if the posterior probability of H_0 is below α, i.e., if $P[\theta\in A \mid X] < \alpha$.

2. Let X have a binomial distribution with n=4 and $p\in\{0,.2,.4,.5,.6,.8,1\}$. Say that the prior distribution of p is given by g(0)=g(1)=.05, g(.2)=.5, and g(.4)=g(.5)=g(.6)=g(.8)=.1. If X=2 is the experimental outcome, find the posterior probability that $p\in\{0,.2,.4\}$ and complete the Bayes hypothesis test of $H_0{:}p\in\{0,.2,.4\}$ versus $H_1{:}p\in\{.5,.6,.8,1\}$. Use $\alpha = 0.50$.

3. Let $X_1, \ldots, X_n, X_{n+1}$ be a random sample from a $N(\mu, \sigma^2)$ population, where $-\infty < \mu < \infty$ and $\sigma^2 > 0$.

(a) Find a constant c such that $T_n^2 = c\sum_{i=1}^{n}(X_{i+1} - X_i)^2$ is an unbiased estimator of σ^2.

(b) Find the maximum likelihood estimators of μ and σ.

(c) Let $Y_1 = X_1 - X_2$, $Y_2 = X_1 + X_2$ and $Y = (Y_1, Y_2)'$. Are Y_1 and Y_2 independent? Explain your reasoning. Furthermore, find the distribution of the bivariate random vector Y.

(d) Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$. Find the distribution of $\frac{X_{n+1} - \bar{X}_n}{S_n}\sqrt{\frac{n-1}{n+1}}$.

4.a. Define: X_n converges in distribution to a random variable Y.

b. Define: X_n converges in probability to a random variable Y.

c. Prove: If X_n converges in distribution to a constant c then X_n converges in probability to c.

d. Let $X_1, X_2, \ldots, X_n$ be independent B(N,p) random variables, with both N and p unknown. Find consistent estimators of both N and p. Prove your assertion. You may use the following: If U_n converges in probability to a and V_n converges in probability to b, with $b \neq 0$, then U_n+V_n converges in probability to a+b, $U_n \times V_n$ converges in probability to $a \times b$, and U_n/V_n converges in probability to a/b.

5. Let $X_1, \ldots, X_n$ be a random sample from a population with density function

$$f(x) = \begin{cases} \frac{2}{\theta} x e^{-\frac{x^2}{\theta}}, & \text{if } x > 0, \\ 0, & \text{elsewhere,} \end{cases}$$

where θ is a positive parameter.

(a) Find the maximum likelihood estimator of θ.

(b) Find the maximum likelihood estimator of θ^2.

(c) Find the distribution of $\frac{2}{\theta} \sum_{i=1}^{n} X_i^2$.

(d) Let $n = 10$. Suppose that we wish to test the null hypothesis $H_0 : \theta = 2$ versus the alternative hypothesis $H_a : \theta = 1$. Use the Neyman-Pearson lemma to find the most powerful critical region of size $\alpha = 0.05$.

6. Let (X,Y) be uniformly distributed on the triangle with vertices at (0,0), (0,a), and (a,0) for some a>0. Further, say that all we can observe is the maximum of the two, say V.

a. Show that the cumulative distribution function (CDF) of V is given by

$$F_V(v) = \begin{cases} \frac{2v^2}{a^2} & \text{if } 0 \le v \le a/2 \\ \frac{2v}{a^2}(2a-v)-1 & \text{if } a/2 \le v \le a \end{cases}$$

and 0 below 0 and 1 above a. Also find the probability density function (pdf) of V.

b. Find the method of moments estimate and the maximum likelihood estimate of a based on V. Also draw a graph of the likelihood function, L(a).

c. Based on one observation of V, find the likelihood ratio test for H_0:a=3 versus H_1:a>3. Find the exact critical region for α=.10.