Department of Mathematics
University of Toledo

Master of Science Degree
Comprehensive Examination
Probability and Statistical Theory

June 24, 1996

Instructions:

Do all four problems.

Show all of your computations in your Blue Book.

Prove all of your assertions or quote appropriate theorems.

Books, notes, and calculators *may be used.*

This is a three hour test.

1. (30) The probability density function of a continuous random variable X is given by

$$f(x) = \begin{cases} cx & \text{if } 0 \le x \le a \\ c(2a-x) & \text{if } a < x \le 2a \\ 0 & \text{otherwise} \end{cases}.$$

a. Find c.

b. Prove that $\mu = E(X) = a$.

c. Show that $\sigma = \text{StDev}(X) = a/\sqrt{6}$.

d. If M(t) denotes the moment generating function of X, find M(0), M'(0), and M"(0).

*** Parts e-g pertain to a random sample $X_1, X_2, \ldots, X_n$ from this distribution. Estimators should use all of the data!!! ***

e. Find the method of moments estimator of a.

f. Find two different unbiased estimators of σ^2.

g. Find an unbiased estimator of σ.

2. (20) Let $X_1, X_2, \ldots, X_n$ denote a random sample from the distribution with density given by

$$f(x) = \frac{1}{\theta} e^{-x/\theta} \text{ for } x \ge 0.$$

a. Find a complete and sufficient statistic for θ.

b. Find the method of moments estimate for θ.

c. Find the maximum likelihood estimate of θ^2.

d. Let $Y = X_1/(X_1+X_2)$. Find E(Y) and Var(Y).

3. (35) Let $X_1, X_2, \ldots, X_n$ denote a random sample from the distribution with density given by

$$f(x) = \frac{\alpha b^{\alpha}}{x^{\alpha+1}} \text{ for } x \ge b, \text{ where } \alpha > 0 \text{ and } b > 0.$$

a. Confirm that this is a density.

b. Find a sufficient statistic for (α, b).

c. Find the method of moments estimates for α and b.

d. Find the maximum likelihood estimates of for α and b.

e. Find the maximum likelihood estimate for α under the restriction that b=1.

f. Find the likelihood ratio test statistic for testing $H_0: b=1$ versus $H_1: b>1$.

g. Suppose that we have the following data (n=20) for performing this test (the top row is the actual observations with statistics at the end and the bottom row gives the natural logs with statistics) :

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                                                                                                  Sample Average: StDev:
x    : 3.99 2.49 3.47 2.65 2.96 4.43 7.10 3.02 3.44 5.78 2.08 3.43 3.29 4.18 3.41 3.03 6.29 4.16 6.14 2.35 -- 3.88  1.41
ln(x): 1.38 0.91 1.24 0.98 1.08 1.49 1.96 1.11 1.24 1.75 0.73 1.23 1.19 1.43 1.23 1.11 1.84 1.43 1.81 0.85 -- 1.30  0.34
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Perform the likelihhod ratio test in part f using the usual large sample approximation. Set α = .05.

4. (15) Let (X,Y) be uniformly distributed over the triangle with vertices at (0,0), (0,a), and (b,0). Find

a. the CDF and pdf of min(X,Y).

b. the method of moments estimators of a and b (based on a random sample from this joint distribution).