Department of Mathematics
University of Toledo

Master of Science Degree
Comprehensive Examination
Probability and Statistical Theory

May 24, 1997

Instructions:

Do FOUR out of five questions.

Clearly state which four you are choosing.

Show all of your computations.

Prove all of your assertions or quote appropriate theorems.

Books, notes, and calculators *may be used.*

This is a three hour test.

1. Consider an experiment in which balls numbered $1, \ldots, n$ are distributed at random in n boxes so that the total number of outcomes is $n!$. Let S_n be the number of matches; i.e., the number of balls in boxes having the same number. Find $\mathrm{E}(S_n)$ and $\mathrm{Var}(S_n)$.

2. Let $X_1, \ldots, X_n$ be a random sample from a uniform $U(0, \theta)$ distribution, where $\theta > 0$.

(a) Are $X_{(n)}, \frac{X_{(n)}}{X_{(n-1)}}, \frac{X_{(n-1)}}{X_{(n-2)}}, \cdots, \frac{X_{(2)}}{X_{(1)}}$ independent? Explain your reasoning.

(b) Find the UMVU estimator of θ^2.

(c) Let $\theta = 1$. Find the distribution of $X_l - X_k$, where $1 \leq k < l \leq n$. In addition, find $\mathrm{E}(X_l - X_k)$ and $\mathrm{Var}(X_l - X_k)$.

3. Let X_t, ... , X_n denote a random sample from the distribution with density

$$f(x;\alpha,\beta) = \alpha e^{\alpha x-\beta} \text{ for } x \leq \beta/\alpha \quad \text{where } \alpha>0.$$

a. Confirm that this is a density.

b. Find a sufficient statistic for (α,β).

c. Find the method of moments estimator (MME) for (α,β).

d. Find the maximum likelihood estimator for α under the condition that $\beta = 0$.

e. Find the maximum likelihood estimator for (α,β) with no restriction placed on β.

(Hint: First show that for each fixed α, the maximum occurs along the line $\beta = \alpha\, x_{(n)}$, where $x_{(n)}$ denotes the maximum of the observations.)

f. Derive the likelihood ratio for testing $H_0:\beta=0$ versus $H_1:\beta<0$.

g. Assume that we have the folowing data from this distribution: n=10 and the ordered observations are {-17,-9,-9,-7,-5,-4, -4, -3, -1, -1}. Find the MME and the MLE for the parameters (α,β). Also perform the large sample (chi-square aaproximation) likelihood ratio test derived i part f.

4. A lake has N fish, N unknown. We will find the MLE for estimating N in the context of the experiment described as follows:

i) Capture and tag 5 fish.
ii) Return these fish to the lake and stir for a while.
iii) Capture 7 fish and let X denote the number of tagged fish in the sample of size 7.

a. What is the distribution of X? Give the formula as a function of N and don't forget the range.

b. Based upon this one observation, give the method of moments estimator for N.

c. Give the likelihood function, L(N;x), again for one observation.

d. Find the set of integers n for which $L(n+1;x) \geq L(n;x)$, i.e., find $\{n\geq 1 : L(n+1;x) \geq L(n;x)\}$.

e. Use your answer in (d) to give a formula for $\hat{N}$, the MLE for N.

5. Let X,Y be Uniformly distributed over A where A is the triangle with vertices at (0,0), (a,0), and (0,b).

a. Find the marginal distributions and the expectations of X and Y.

b. Find the CDF and PDF of S = X + Y.

c. Based on a sample of size n from this joint distribution, give the method of moments estimator of the parameter vector (a,b).