Department of Mathematics
University of Toledo

Master of Science Degree
Comprehensive Examination
Probability and Statistical Theory

April 11, 1998

Instructions:

Do all five problems.

Show all of your computations in your Blue Book.

Prove all of your assertions or quote appropriate theorems.

Books, notes, and calculators *may be used.*

This is a three hour test.

Points --
1 - 10
2 - 20
3 - 20
4 - 25
5 - 25

The Examination Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.

1. An urn contains three red and two white balls. A ball is drawn, and then it and another ball of the same color are placed back in the urn. Finally, a second ball is drawn.

(a) What is the probability that the second ball drawn is white?

(b) If the second ball drawn is white, what is the probability that the first ball drawn was red?

2. Let

$$Y_1 = \beta_1 + \epsilon_2$$

$$Y_2 = 2\beta_1 - \beta_2 + \epsilon_2$$

$$Y_3 = \beta_1 + 2\beta_2 + +\epsilon_3,$$

where ϵ_1, ϵ_2, and ϵ_3 are uncorrelated random variables with $E(\epsilon_i) = 0$ and $\text{Var}(\epsilon_i) = \sigma^2$ for $i = 1, 2, 3$.

(a) Find the least squares estimates $(\hat{\beta}_1, \hat{\beta}_2)$ of (β_1, β_2).

(b) Find $\text{Var}(\hat{\beta}_1)$ and $\text{Var}(\hat{\beta}_2)$.

(c) Find $\text{Cov}(\hat{\beta}_1, \hat{\beta}_2)$.

(d) Suppose further that $\sigma^2 = 1$ and $\epsilon_1, \epsilon_2, \epsilon_3$ are iid $N(0, 1)$ random variables. Find a complete and sufficient statistic for (β_1, β_2).

3. Let $X_1, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ population, where $-\infty < \mu < \infty$ and $\sigma^2 > 0$.

(a) Find a complete and sufficient statistic for (μ, σ^2).

(b) Let $\bar{X} = \frac{1}{n}\sum_{i=1} X_i$ and $S_n^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2$. Find $E(\bar{X}^3 S_n^4)$.

(c) If σ^2 is known, find the maximum likelihood estimate of μ^3.

(d) If σ^2 is known, find the UMVU estimator of μ^3. Does the UMVU estimator achieve the Cramér-Rao lower bound?

4. Let Y_1, Y_2 and Y_3 denote three observations from the exponential distribution with mean 1, and let $X_i = Y_i + \theta$ for some $\theta \geq 0$.

a. Find the CDF of the first order statistic, $X_{(1)}$ (i.e., the minimum value).
b. Find the pdf, of $X_{(1)}$.
c. Under the constraint given for θ, find the method of moments estimator for θ based on X_1, X_2 and X_3.
d. Similarly, under this constraint, find the maximum likelihood estimator for θ based on this sample.
e. Find the form of the likelihood ratio test for H_0: $\theta = 0$ versus H_A: $\theta > 0$.
f. For α=0.10, find the exact critical region for the likelihood ratio test in part f.

15. Let X_1 and X_2 be U[0,1] and let N be B(2,p) with all three independent. Further, let $S = \sum_{i=1}^{N} X_i$, that is, S is either 0, X_1, or $X_1 + X_2$, depending on the value of N. In our experiment, we only observe S.

a. Find the distribution of the random variable $U = E(S|N)$. Also find $E(U)$.
b. Find the conditional distribution of S given N=n for n=0,1,2. *Answer* --

$$\text{if } 0 \leq s \leq 2, \text{ then } P(S \leq s \mid N=0) = 1$$

$$\begin{aligned} &\text{if } 0 \leq s \leq 1, \text{ then } P(S \leq s \mid N=1) = s \\ &\text{if } 1 < s \leq 2, \text{ then } P(S \leq s \mid N=1) = 1 \end{aligned}$$

$$\begin{aligned} &\text{if } 0 \leq s \leq 1, \text{ then } P(S \leq s \mid N=2) = s^2/2 \\ &\text{if } 1 < s \leq 2, \text{ then } P(S \leq s \mid N=2) = -1+2s-s^2/2 \end{aligned}$$

c. Find the CDF of S. Note that S is partly discrete.
d. Find the pdf of S for s>0. *Answer* --

$$f(s) = \begin{cases} 2p(1-p) + p^2 s & \text{if } 0<s\leq 1 \\ p^2(2-s) & \text{if } 1<s\leq 2 \end{cases}$$

e. Find E(S). This should match something in part a. What? Does it match?
f. Find the method of moments estimator (MME) for p based on S.
g. Find the maximum likelihood estimator (MLE) for p based on S.