Department of Mathematics
University of Toledo

Master of Science Degree
Comprehensive Examination
Probability and Statistical Theory

April 10, 1999

Instructions:

Do all 4 questions.
Show all of your computations.
Prove all of your assertions or quote appropriate theorems.
Books, notes, and calculators *may be used.*
This is a three hour test.

1. Let X, Y, and Z be independent random variables with each having a uniform distribution $U[0,1]$ on $[0,1]$. Let $M = \max(X, Y, Z)$.

(a) Find $P(Z \geq XY)$.

(b) Find the probability density function of M.

(c) Find $E(M)$ and $\text{Var}(M)$.

2. Let $X_1, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ population, where $-\infty < \mu < \infty$ and $\sigma^2 > 0$.

(a) Find a complete and sufficient statistic for (μ, σ^2).

(b) Let $\bar{X} = \frac{1}{n}\sum_{i=1} X_i$ and $S_n^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2$. Find $\text{Var}(\bar{X}^2 + S_n^2)$.

(c) If σ^2 is known, find the maximum likelihood estimator of $\mu(1-\mu)$.

(d) If σ^2 is known, find the UMVU estimator of $\mu(1-\mu)$. Does the UMVU estimator achieve the Cramér-Rao lower bound?

(e) Use the measure of *mean square error* to compare the maximum likelihood estimator in part (c) with the UMVU estimator in part (d). Which estimator is better? Explain your reasoning.

3. Let X_1, X_2, X_3 be i.i.d. exponential with mean θ, i.e., they denote a random sample from the distribution with density

$$f(x;\theta) = e^{-x/\theta}/\theta \text{ for } x \geq 0 \quad \text{where } \theta > 0.$$

Further, let $X_{(1)} \leq X_{(2)} \leq X_{(3)}$ denote the order statistics of this sample.

a. Find the MLE for θ. Compute its expectation and variance. What are its bias and root mean square error (RMSE)?

b. Find the distribution, expectation and variance of the sample median, $X_{(2)}$.

c. Construct an unbiased estimator for θ which is a linear function of $X_{(2)}$. Find its RMSE and compare this estimator to the estimator from part a.

d. Derive the likelihood ratio test of H_0: $\theta = 1$ versus H_A: $\theta > 1$. Here are some steps to consider:

1) Find and roughly sketch the likelihood function, recalling that its domain is limited to $\theta \geq 1$.

2) Beware in your sketch that there are two important cases, depending on the value of $\bar{x}$.

3) Compute the likelihood ratio λ for both cases.

e. Even though n=3, perform the large sample (chi-square approximation) likelihood ratio test for the data {1.7, 2.1, 1.5}.

4. Assume that we have a joint density for the random vector (X,Y) given by

$$f(x,y) = ky \text{ for } (x,y) \text{ in the region bounded by } y=0,\ x=\theta, \text{ and } y=x^2,$$
$$\text{that is, for } 0 \leq x \leq \theta \text{ and } 0 \leq y \leq x^2.$$

a. Find k.

b. Find the density and expectation of Y.

c. Say that when we actually do the experiment, we can only observe. Y and not X. If, for n=2, we observe Y = 1 and Y = 4, find the MLE for θ^2 based on this information.

d. Also find the method of moments estimator for θ^2 based on this information.