Ph.D. Qualifying Examination
Fall 2003

Instructions:

1. If you think that there is a mistake ask the proctor. If the proctor's explanation is not satisfactory, interpret the problem as you see fit, but not in such a way that the answer is trivial.
2. From each part solve 3 of 6 problems.
3. If you solve more that three problems from a part indicate the problems that you wish to have graded.

Part A: ODE Questions

1. Consider the autonomous system $\dot{x} = f(x)$ where $f : R^n \to R^n$ is Lipshitz continuous. Suppose that $f(x)$ satisfies $< x, f(x) > \geq \ ||x||^3$. Show that the solution to the initial value problem $x(t_0) = x_0 \neq 0$ cannot extend to $[t_0, \infty)$.

2. Suppose that $g(v)$ is a continuously differentiable function of a single variable satisfying $0 < g'(v) \leq 2M$ and $0 < g(v) < Mv$ for some positive constant M and $v > 0$. Consider the initial value problem

$$\dot{v} = h + g(v); \quad v(0) = 1$$

where h is a constant. Show that the solution $v(t)$ satisfies

$$|v(t) - (ht + 1)| \leq \frac{1}{2}(\frac{h}{M} + 1)e^{2Mt}.$$

3. Find the fundamental solution of $X(t)$ with $X(0) = I$ to the system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_2 + x_3 \\ x_1 + x_4 \\ -x_4 \\ x_3 \end{pmatrix}.$$

4. Consider the system $\dot{x} = h(t)Ax$ where $x : R \to R^n$, A is a constant $n \times n$ matrix, and $h(t)$ is strictly positive and continuous. Show that 0 is

asymptotically stable if all the eigenvalues of A have negative real parts and $\int_0^\infty h(t)dt$ diverges. By example show that the stability may not hold if the integral converges.

5. Let L be a periodic solution of a Lipschitz continuous planar autonomous system $\dot{x} = f(x)$ with flow $\phi_t : R^2 \to R^2$. Recall that an $\epsilon-$ neighborhood N_ϵ of L is small if N_ϵ contains no singular points and for any $q_1, q_2 \in N_\epsilon$ with $|q_1 - q_2| < 2\epsilon$,

$$\frac{< f(q_1), f(q_2) >}{||f(q_1)||\,||f(q_2)||} > \frac{1}{\sqrt{2}}.$$

Suppose that $L^1 \subset N_\epsilon$ is a second periodic solution and that for some $p \in L$ and tranverse segment $\overline{n^1pn}$ to L there is a $p^1 \in L^1$ with $p^1 \in \overline{n^1pn}$. Let T be the smallest parameter value so that $\phi_T(p^1) \in \overline{n^1pn}$. Show that T is the period of L^1.

6. Suppose $p(t), q(t)$ are continuous functions on an interval I. Let $y_1(t), y_2(t)$ be two linearly independent solutions of

$$y'' + p(t)y' + q(t)y = 0.$$

Prove that between any two roots of $y_1(t)$ there is at least one root of $y_2(t)$.

Part B: PDE Questions

1. Consider the quasi-linear system for an unknown function $u : R^n \times R \to R$ given by

$$\frac{\partial u}{\partial t} + \vec{a}(\vec{x}, t) \cdot \frac{\partial u}{\partial \vec{x}} = b(\vec{x}, t, u)$$

where $t \in R$ and $\vec{x} \in R^n$. Show that if $\vec{a}, b$ are bounded and Lipshitz continuous, then the Cauchy problem with Cauchy data defined on the hyperplane $H = \{(\vec{x}, t) \,|\, t = 0\}$ has global solutions.

2. Determine the canonical form and the general solution to the second order euqation

$$4x^2u_{xx} + 4xyu_{xy} - 8y^2u_{yy} + 4xu_x - 8u_y = 0.$$

3. Consider the Cauchy-Kowalewski system

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} v & 0 \\ 0 & -u \end{pmatrix}\frac{\partial}{\partial y}\begin{pmatrix} u \\ v \end{pmatrix}.$$

Determine the Taylor expansion of the solution to second order with Cauchy data given by $u(x, y, 0) = x, v(x, y, 0) = y$.

4. Suppose that on a bounded domain Ω a function $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies $\Delta u + \lambda u^3 = 0$ in Ω with $\lambda > 0$ and $u = 0$ on $\partial\Omega$. Suppose that u is non-negative in Ω. Show that u is strictly positive in Ω.

5. Let u be a harmonic function on a bounded domain Ω. Suppose $\phi : R \to R$ is a convex C^2 function. Show that $\phi \circ u$ is subharmonic.

6. Suppose w is a harmonic function on R^n and satisfies

$$\int_{R^n} w^2(x)dx < \infty.$$

Prove that $w \equiv 0$.