Fall 2003 Ph.D Qualifying Exam in Real Analysis

Time 3 hours, closed book, no notes. Answer three questions from each of the parts A & B.

Part A

1. (The Ascoli-Arzela Theorem) Let X be a compact metric space and let $C(X)$ be the normed linear space of real-valued continuous functions on X with the supremum norm.
   (a) Define what it means for a subset $\mathcal{F} \subset C(X)$ to be an equi-continuous family.
   (b) Prove that $\mathcal{F}$ is totally bounded in $C(X)$ if and only if
      i. $\mathcal{F}$ is bounded and
      ii. $\mathcal{F}$ is an equi-continuous family.

2. (a) Consider the power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$ where the sequence $\{a_k\}$ is bounded. Show
      i. Both series converge uniformly on $[-\rho, \rho]$ for any ρ, $0 < \rho < 1$,
      ii. $f'(x) = g(x)$ for $-1 < x < 1$.
   (b) Suppose $0 \leq k \leq |a_k| \leq k^2$ for all k. What is the radius of convergence of the power series $\sum_{k=0}^{\infty} a_k x^k$?

3. Suppose (X, d) is a compact metric space and that f is an isometry of X into itself(i.e., $f : X \to X$ with $d(f(x_1), f(x_2)) = d(x_1, x_2)$ for x_1, x_2 in X.) Clearly f is a 1–1 map(an injection). Show that f is an onto map(a surjection).[Hint: Take a point p outside $f(X)$ if possible and look at the sequence of iterates $p, f(p), f(f(p)), \ldots$.]

4. Let $f(x)$ be continuous on $[0, 1]$. Show
   (a) $\lim_{n\to\infty} \int_0^1 x^n f(x) dx = 0$,
   (b) $\lim_{n\to\infty} \int_0^1 n x^n f(x) dx = f(1)$.

5. Prove Egoroff's Theorem namely: Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $\mu(X) < \infty$. Let $\{f_n\}$ be a sequence of measurable functions such that $f_n \to f$ (a.e.). Show that for any $\epsilon > 0$, there is a measurable set E with $\mu(E) < \epsilon$ such that $f_n \to f$ uniformly on $X \backslash E$.

6. Let $\phi_n(x) = \sqrt{n} e^{-n^2|x|}$ and let $f(x) = \sum_{n=0}^{\infty} \phi_n(x - r_n)$ where $\{r_n\}$ is an enumeration of rational numbers in $\mathbb{R}$. Show:
   (a) $f(x) \in L^1(\mathbb{R})$ and compute $\int_{\mathbb{R}} f(x) dx$
   (b) $f(x)$ is unbounded in any open interval (a, b).

Part B

1. Let $f(x) = 0$ if $x \in [0,1]$ is irrational and $f(x) = \dfrac{1}{q}$ if $x = \dfrac{p}{q}$ in lowest terms. Show: $f(x)$ is continuous at every irrational point and discontinuous at every rational point in $[0,1]$.

2. Let $f_n(x) = (1+x^n)^{1/n}$, $0 \leq x \leq 2$, $n \in \mathbb{N}$. Show that $f_n(x) \rightrightarrows f(x)$ uniformly on $[0,2]$ as $n \to \infty$ where $f(x) = 1$ for $x \in [0,1]$ and x for $x \in [1,2]$.

3. (a) State carefully the Stone-Weierstrass Theorem.
   (b) Prove that if $f(x)$ is continuous in $[0,1]$ and if $\int_0^1 x^{4n} f(x)dx = 0$ for $n = 0, 1, 2, \ldots$, then $f(x) \equiv 0$.
   (c) Suppose $\int_0^1 x^k f(x)dx = 0$ for k odd. What can you say about $f(x)$?

4. Show that

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n dx = \lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} dx = 1.$$

5. Suppose that f, a function defined on an open interval (a,b), satisfies the intermediate value theorem i.e., if f assumes the values y_1, y_2, it assumes all values between y_1, y_2. Show that if f is not continuous, it assumes some value infinitely often.

6. State the definition of Riemann integrability and prove directly that any continuous function on a closed interval $[a,b]$ is Riemann integrable.