Ph.D. Qualifying Exam

Fall 2004

Instructions:

1. If you think that a problem is incorrectly stated ask the proctor. If his explanation is not to your satisfaction, interpret the problem as you see fit, but not so that the answer is trivial.

2. From each part solve 3 of the 5 five problems.

3. If you solve more than three problems from each part, indicate the problems that you wish to have graded.

Part A

1. Suppose that $a_n > 0$, show that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges.

2. Let A be a closed and bounded subset of $C_2([0,1])$, the twice continuously differentiable functions with the supremum norm . Show that A is precompact subset of $C_1([0,1])$.

3. Consider the closed unit ball B in $C_0([0,1])$, show that B cannot be covered by a finite number of balls of radius r where $r < 1$.

4. Suppose that $f_n(x)$ is a decreasing sequence of upper semi-continuous function with pointwise limit $f(x)$. Show that $f(x)$ is upper semi-continuous.

5. If $\sum a_n$ is a divergent series of positive terms and s_n denotes its n^{th} partial sum, show that the series $\sum \frac{a_n}{s_n^2}$ converges.

Part B

1. Consider the sequence of functions $f_n(x) = \dfrac{ne^{x^2}}{1+n^2x^2}$. Compute $\lim_{n\to\infty} \int_0^1 f_n(x)dx$. Justify each of your steps carefully.

2. Suppose that $E \subset \mathbf{R}$ is Lebesgue measurable and that there is a real number α, $0 \le \alpha < 1$ such that for any interval I, $\mu(E \cap I) \le \alpha\mu(I)$ where μ is Lebesgue measure. Show that $\mu(E) = 0$.

3. Suppose that $\{f_n(x)\}$ is a sequence of real valued measurable functions of a real variable. Suppose that there is an integrable function g with $|f_n(x)| < g(x)$ for all n. Show that $\limsup \int f_n(x)dx \leq \int \limsup f_n(x)dx$.

4. Suppose that a sequence of real valued measurable function $\{f_n(x)\}$ converges in measure to a function $f(x)$, and there is an integrable function g with $|f_n(x)| < g(x)$. Show $\{f_n(x)\}$ converges to $f(x)$ in mean.

5. Let $\{f_n(x)\}$ be a sequence of real valued measurable functions. Show that the set where $\{f_n(x)\}$ converges is measurable.