Ph.D. Qualifying Exam
ALGEBRA

University of Toledo
Department of Mathematics
January 21, 2006

Instructions

Do any four problems. And no more than four.
Please make sure that you give complete solutions and full explanations to each problem that you do.

Indicate which problems you wish to have graded.

You have three hours.

Policy on Misprints

The Ph.D. Qualifying Examination Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.

1. For each of the following either give an example or else prove that no such example is possible.

   (a) A nonabelian group.

   (b) A finite abelian group that is not cyclic.

   (c) An infinite group with a subgroup of index 5.

   (d) Two finite groups that have the same order but are not isomorphic.

   (e) A group G with a subgroup H that is not normal.

   (f) A nonabelian group with no normal subgroups except the whole group and the unit element.

   (g) A group G with a normal subgroup H such that the factor group G/H is not isomorphic to any subgroup of G.

   (h) A group G with a subgroup H which has index 2 but is not normal.

2. Let $\mathbf{F} \subset \mathbf{K}$ be fields, and a and b elements of $\mathbf{K}$ which are algebraic over $\mathbf{F}$. Show that $a + b$ is algebraic over $\mathbf{F}$.

3. Let A denote the ideal in $\mathbb{Z}[x]$, the ring of polynomials with coefficients in $\mathbb{Z}$, generated by $x^3 + x + 1$ and 5. Is A a prime ideal?

4. Let R be a principal ideal domain and let A and B be nonzero ideals in R. Show that $AB = A \cap B$ if and only if $A + B = R$.

5. Let $\mathbb{Z}^2$ be the group of lattice points in the plane (ordered pairs of integers, with coordinatewise addition as the group operation). Let H_1 be the subgroup generated by the two elements $(1,2)$ and $(4,1)$, and let H_2 the subgroup generated by the two elements $(3,2)$ and $(1,3)$. Are the quotient groups $G_1 = \mathbb{Z}^2/H_1$ and $G_2 = \mathbb{Z}^2/H_2$ isomorphic?

6. Suppose that R is a subring of a commutative ring S and that R has finite index n in S. Let m be an integer that is relatively prime to n. Prove that the natural map $R/mR \to S/mS$ is a ring isomorphism.

7. Let p, q, r and s be polynomials of degree at most 3. Which, if any, of the following two conditions is sufficient for the conclusion that the polynomials are linearly dependent?

   (a) At 1 each of the polynomials has the value 0.

   (b) At 0 each of the polynomials has the value 1.

8. Are the matrices given below similar?

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$