Differential Equations

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Do three questions from both Part I and from Part II

I. Ordinary Differential Equations

DO THREE of the following 5 problems.

1. (a) Solve the initial value problem for the differential equation.

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{for general} \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

   (b) Sketch a few sample trajectories.

2. Consider the initial value problem

$$\mathbf{x}'(t) = \mathbf{F}(t, \mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

where $|t - t_0| < T$ for some $T > 0$ and $\mathbf{x}_0 \in \mathbb{R}^n$. Suppose that $\mathbf{F}(t, \mathbf{x})$ is continuous in $(t, \mathbf{x})$ for all $|t - T_0| < T$ and $\mathbf{x} \in \mathbb{R}^n$ and satisfies the Lipschitz condition

$$|\mathbf{F}(t, \mathbf{x}) - \mathbf{F}(t, \mathbf{y})| < L|\mathbf{x} - \mathbf{y}|$$

for some $L > 0$. Show that there exists a solution $\mathbf{x}(t)$ of the given initial value problem defined in a neighborhood of t_0.

3. Again consider the initial value problem.

$$\mathbf{x}'(t) = \mathbf{F}(t, \mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

where $\mathbf{F}$ is as in the previous question, that is $\mathbf{F}(t, \mathbf{x})$ is continuous and is Lipschitz in $\mathbf{x}$. Show that, if any solution exists, then it must be unique.

4. Let y_1 and y_2 be two linearly independent solutions of the differential equation

$$y'' + p(x)y' + q(x)y = 0$$

where $p(x)$ and $q(x)$ are continuous real valued functions.

   (a) Show that the zeroes of y_1 are isolated (that is $\{x : y_1(x) = 0\}$ has no accumulation points).

   (b) Show that the Wronskian $W(y_1, y_2)(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$ is never zero.

   (c) Show that, between any two successive zeroes of y_1, there is exactly one zero of y_2.

5. Find the solution of the initial value problem $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \quad \text{where} \quad \mathbf{x}(0) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Remark: A is in "Jordan canonical form."

II. Partial Differential Equations

DO THREE of the following 5 problems.

1. Solve $U = xU_x + yU_y + ({U_x}^2 + {U_y}^2)/3$ with initial condition $U(x,0) = (1-2x^2)/2$.

2. Solve the initial value problem and **verify** your solution

$$uu_x + yu_y = x, \quad u(x,1) = 2x.$$

3. Suppose that $u(x)$ is a solution of

$$\Delta u + au_x + bu_y = cu$$

which is twice continuously differentiable on the open unit disk $D = \{(x,y) : x^2 + y^2 < 1\}$ and continuous on the closure $\overline{D}$ of D. Suppose that $a = a(x,y)$, $b = b(x,y)$ and $c(x,y)$ are continuous and $c(x,y) > 0$ on $\overline{D}$. Show that, if $u = 0$ on the boundary δD, then $u \equiv 0$ on D. [Suggestion: Show that $\max u \leq 0$ and $\min u \geq 0$.]

4. Solve the following initial value problem.

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}_t + \begin{pmatrix} 2 & 8 \\ 8 & 2 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$U_1(x,0) = \sin x$$

$$U_2(x,0) = \cos x$$

5. Solve the initial/boundary value problem

$$u_{tt} - u_{xx} = 1 \quad for \quad 0 < x < \pi \quad and \quad t > 0$$

$$u(x,0) = 0, \quad u_t(x,0) = 0 \quad for \quad 0 < x < \pi$$

$$u(0,t) = 0, \quad u(\pi,t) = -\pi^2/2 \quad for \quad t \geq 0.$$