Ph.D. Qualifying Exam: Real Analysis

January 13, 2007

Instructions: Do six of the 9 questions. No materials allowed.
Examiners: Rao Nagisetty; Denis White.

1. Let $(\Omega, \mathcal{F}, \mu)$ be a measurable space and $f_n : \Omega \to \mathbb{R}$ be a sequence of measurable, real valued functions. If f_n converges pointwise to a function f then show that f is measurable.

2. Give an example of a sequence of functions f_n defined and Riemann integrable on $[0, 1]$, such that $|f_n| \le 1$, for all $n \in \mathbb{N}$ and $f_n \to f$ pointwise everywhere but f is not Riemann integrable. Is f necessarily Lebesgue integrable? Explain.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function. Show that

$$\lim_{t \to \infty} \int_{\mathbb{R}} f(x) \sin(xt) dx = 0.$$

4. (a) State the Baire Category Theorem for Complete Metric spaces.

   (b) If $\{f_n\}$ is a sequence of real valued continuous functions converging pointwise to finite valued function f on a non-empty complete metric space X, show that given any $\epsilon > 0$, there exists a non-empty open set V and a positive integer N such that $|f_n(x) - f(x)| < \epsilon$ for all x in V and all $n > N$.

5. (a) State the Stone-Weierstrass Theorem.

   (b) Let $C([0, 1])$ denote the space of continuous functions on the closed interval [0,1] with the usual "sup norm" topology. Prove or disprove that the vector space generated by $\{1, x^2, x^4, \ldots, x^{2n}, \ldots\}$ is dense in $C([0, 1])$.

   (c) Prove or disprove that the vector space generated by $\{1\}$ and $\{x^{an+b} : n \in \mathbb{N}\}$ where a,b are fixed positive integers is dense in $C([0, 1])$.

6. (a) State the Ascoli-Arzela's Theorem.

   (b) Let S be the set of all continuously differentiable functions on the interval $[0,1]$ such that $f(0)=1, |f'(x)| \leq 1$ *on* $[0,1]$. Show that S is a compact subset of the space of all continuous functions on the interval $[0,1]$.

   (c) Let L be the set of all twice continuously differentiable functions on the interval $[0,1]$ such that $f(1/2)=0, f'(0)=1, |f''(x)| \leq 12$. Prove or disprove L is a compact subset of the space of all continuous functions on the interval $[0,1]$.

7. (a) Define a convex function on an open interval of the real line.

   (b) Show that any convex function is continuous.

   (c) Show that any convex function is either decreasing or increasing or initially decreasing but eventually increasing.

8. (a) Define a function of bounded variation on the interval $[0,1]$.

   (b) Show that any function of bounded variation is a difference of two increasing positive functions.

   (c) Show that the function $f(x)$ defined as $f(x)=\sin(1/x)$ *for* $x > 0, f(0)=1$ is not of bounded variation in $[0,1]$.

9. (a) Euler's Γ function is defined by $\Gamma(a)=\int_0^\infty x^{a-1}e^{-x}\,dx$ for $a>0$. Show that $\Gamma(a+1)=a\Gamma(a)$ and $\lim_{a\to 0+} a\Gamma(a)=\Gamma(1)=1$.

   (b) Show that the function $f(x)=\dfrac{e^{-x}-e^{-3x}}{x}$ is summable in the interval $[0,\infty)$.

   (c) Evaluate the integral $\int_0^\infty f(x)\,dx$ where f is as in Part (b) above.

   Suggestion: Evaluate $\lim_{a\to 0+}\int_0^\infty x^a f(x)\,dx$ using Part (a).