Topology Ph.D. Qualifying Exam

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This examination has been checked carefully for errors. If you find what you believe to be an error in a question, report this to the proctor. If the proctor's interpretation still seems unsatisfactory to you, you may modify the question so that in your view it is correctly stated, but not in such a way that it becomes trivial. If you feel that the examination is on the long side do not panic. The grading will be adjusted accordingly.

1 Part One: Do six questions

1. True or false: if (X, d) is a metric space then d' defined by $d(x, y) = \frac{d(x,y)}{3+d'(x,y)}$, $(x, y \in X)$ is also a metric on X. Explain.

2. Let C be an closed subset of a topological space X. Prove that a subset $A \subset C$ is relatively closed in C if and only if A is closed in X.

3. Define compactness for a topological space. A collection of subsets is said to have the finite intersection property if every finite class of those subsets has empty intersection. Prove that a topological space is compact iff every collection of closed subsets that has the finite intersection property itself has empty intersection.

4. Let γ be a given cover of a topological space X. Assume that for each member $A \in \gamma$, there is given a continuous map $f_A : A \to Y$ such that

$$f_A \mid A \cap B = f_B \mid A \cap B$$

for each pair of members A and B of γ. Then we may define a function $f : X \to Y$ by taking

$$f(x) = f_A(x), \qquad (\text{if } x \in A \in \gamma).$$

Prove that if γ is a finite closed cover of X, then the function f is continuous.

5. Let $X = \Pi_{\mu \in M} X_\mu$ be the product of the topological spaces $(X_\mu)_{\mu \in M}$ and with X having the product topology. Prove that the projection $p_\mu : X \to X_\mu$ is an open map from X onto X_μ for each $\mu \in M$.

6. Prove that a path-connected topological space is connected.

7. Prove that if two connected sets A and B in a space X have a common point p, then $A \cup B$ is connected.

8. Prove that every compact set K in a Hausdorff space X is closed.

9. A topological space is said to be *locally compact* if every point has a compact neighborhood. Prove that every closed subspace of a locally compact space is locally compact.

10. Let X be a topological space. Let $A \subset X$ be connected. Prove that the closure $\overline{A}$ of A is connected.

11. Prove that if $f : X \mapsto Y$ is continuous and surjective and X is compact and Y Hausdorff then f is an identification map.

12. Prove or disprove: in a compact topological space every infinite set has a limit point. If you cannot answer the question for a compact topological space answer it for a metric space.

2 Part Two: Do three questions

1. Let
$$S^1 = \{(x, y)|x^2 + y^2 = 1\}$$
and
$$S^4 = \{(x_1, x_2, x_3, x_4, x_5)|x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 1\}.$$
(i) Let A be the antipodal map on S^1 defined by $A(x, y) = (-x, -y)$. Show that A is homotopic to the identity map on S^1.

(ii) Let B be the map on S^4 defined by $B(x_1, x_2, x_3, x_4, x_5) = (-x_1, -x_2, x_3, -x_4, -x_5)$. Show that B is homotopic to the identity map on S^4.

2. Let X and Y be topological spaces.

(i) Define what it means for X and Y to have the same homotopy type.

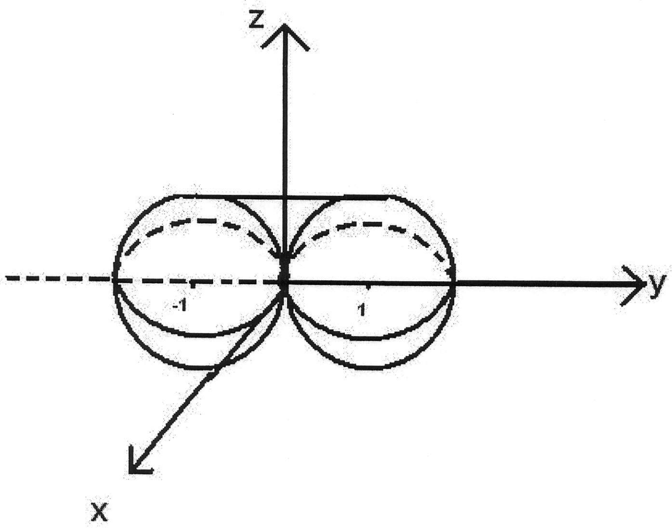
(ii) A space is contractible if it is homotopy equivalent to the one-point space. Prove that X is contractible if and only if the identity map $id_X : X \mapsto X$ is homotopic to a map $r : X \mapsto X$ whose image is a single point.

(iii) Suppose that $Y \subset X$. Define what it means for Y to be a retract of X.

(vi) Prove that a retract of a contractible space is contractible.

3. (i) Let S^2 be a two dimensional sphere. What is $\pi_1(S^2)$? Please explain your answer.

(ii) Let $X = X_1 \cup X_2 \cup X_3$ where
$$X_1 = \{(x, y, z) \in R^3|x^2 + (y - 1)^2 + z^2 = 1\},$$
$$X_2 = \{(x, y, z) \in R^3|x^2 + (y + 1)^2 + z^2 = 1\}$$
and
$$X_3 = \{(0, y, 1)| - 1 \le y \le 1\}.$$
The graph of X is sketched below. Find $\pi_1(X)$.



4. Let P^2 be the two-dimensional real projective space and T^2 be the two-dimensional torus.

   (i) What is $\pi_1(P^2)$? Explain your answer.

   (ii) The space P^2 can be obtained from the disk D^2 by identifying $x \sim -x$ if $||x|| = 1$. Let $p \in \text{int}(D^2)$, the interior of D^2. Find the fundamental group of $P^2 - \{[p]\}$.

   (iii) Let $f : P^2 \mapsto T^2$ be a continuous map. Show that f is null homotopic.

5. It is well-known that the punctured plane $\mathbb{R}^2 - (0,0)$ has the structure of a topological group with multiplication defined by $(x,y) \cdot (p,q) = (xp - yq, xq + yp)$ (induced by multiplication of complex numbers). Can one define a topological group structure on $\mathbb{R}^2 - \{(1,0), (-1,0)\}$? Explain.

6. Outline the main points in the construction of the fundamental group of a topological space.