University of Toledo Department of Mathematics
Ph.D. Qualifying Exam in Algebra
Jan 26, 2008

Instructions: Please do *six* problems, including *at least one problem from each of the three sections*. Give complete proofs. If you attempt more than six problems, indicate clearly which six problems you would like graded. You have three hours.

1. Groups

(1) Prove that a group of order 825 is solvable.

(2) Let G be a group of order 2008. Prove that G contains an abelian subgroup of index 2.

(3) Prove that there is no simple group of order 90.

(4) Let H and K be subgroups of a finite group G. Suppose that $|G : H| = p$, where p is a prime and p is strictly smaller than every prime divisor of $|K|$. Prove that K is a subgroup of H.

2. Fields

(5) Let $K \subseteq F$ be a field extension. If $0 \neq b \in F$, show that b is algebraic over K if and only if b^{-1} is a polynomial in b with coefficients from K.

(6) Find the Galois group of the splitting field over $\mathbb{Q}$ (the rational numbers) of the polynomial $x^4 - 14x^2 + 9$.

(7) Let E/K be a field extension of degree p, where p is a prime. Suppose $f(x) \in K[x]$ is an irreducible polynomial which has more than one root in E. Prove that $f(x)$ splits in E.

(8) If F is a *finite* field, show that each element of F is a sum of two squares. (*Hint:* Consider the set of squares in F and the map $x \mapsto x^2$.)

(9) Let R be a non-zero commutative ring and let M be a non-zero torsion-free left R-module. (M is said to be *torsion-free* if for any non-zero $m \in M$, if $r \in R$ such that $rm = 0$ then $r = 0$.) Show that there is an R-monomorphism from M into a vector space over a field. (*Hint:* What if $M = R$?)

(10) Suppose that P_1, P_2 and P_3 are ideals of a commutative ring R and that P_1 is a prime ideal. If S is a subring of R and $S \subseteq P_1 \cup P_2 \cup P_3$, show that $S \subseteq P_i$ for some i.

(11) Let R be a non-zero ring with the property that each ascending sequence of ideals in R is constant after finitely many terms.

(a) If $f : R \to R$ is a surjective ring homomorphism, prove that f is an isomorphism.

(b) Show that the rings R and $R[x]$ are not isomorphic.

(c) Give an example to show that (b) can fail if the condition on ascending sequences of ideals is not satisfied.

(12) Let R be a ring and let V be a right R-module. Assume that $M_1, M_2, \ldots, M_n$ are finitely many R-submodules of V such that $M_1 \cap M_2 \cap \ldots \cap M_n = 0$, and let W be the (external) direct sum $W = V/M_1 \oplus V/M_2 \oplus \ldots \oplus V/M_n$.

(a) Show that V is isomorphic to an R-submodule of W.

(b) Suppose in addition that the modules V/M_i are simple and pairwise nonisomorphic. Prove that V is isomorphic to W.