Department of Mathematics
The University of Toledo

Ph.D. Qualifying Examination
Probability and Statistical Theory

January 26, 2008

Instructions

Do all four problems.

Show all of your computations.
Prove all of your assertions or quote appropriate theorems.
This is a closed book examination.
This is a three hour test.

1. Let Ω be a set, $\mathcal{F}$ be σ-field on Ω, and $C \in \mathcal{F}$. Show that $\mathcal{F}_C = \{C \cap A : A \in \mathcal{F}\}$ is a σ-field on C.

2. Let $(\mathcal{X}, \mathcal{B}, \{P_\theta : \theta \in \Theta\})$ be the statistical space associated with the random variable X, where $\mathcal{B}$ is the σ-field of Borel subsets $A \subset \mathcal{X}$ and $\{P_\theta : \theta \in \Theta\}$ is a family of probability distributions defined on the measurable space $(\mathcal{X}, \mathcal{B})$ with Θ an open subset of $\mathbf{R}^k$. We assume that the probability distributions P_θ are absolutely continuous with respect to a σ-finite measure μ on $(\mathcal{X}, \mathcal{B})$. Let

$$f(x;\theta) = \frac{dP_\theta(x)}{d\mu(x)}$$

denote the family of probability density functions. The power-divergence measure between the probability distributions P_{θ_1} and P_{θ_2} is defined by

$$I_\lambda(\theta_1, \theta_2) = \frac{1}{\lambda(\lambda+1)}\left(\int_{\mathcal{X}} \frac{f^{\lambda+1}(x,\theta_1)}{f^\lambda(x,\theta_2)} d\mu(x) - 1\right), \qquad \lambda \in \mathbf{R}, \quad \lambda \neq 0, 1.$$

(a) Under suitable regularity conditions, find

$$I_0(\theta_1, \theta_2) = \lim_{\lambda \to 0} I_\lambda(\theta_1, \theta_2).$$

(b) Under suitable regularity conditions, find

$$I_{-1}(\theta_1, \theta_2) = \lim_{\lambda \to -1} I_\lambda(\theta_1, \theta_2).$$

(c) Show that $I_{-1}(\theta_1, \theta_2) = I_0(\theta_2, \theta_1)$.

3. Let $X_1, \cdots, X_n$ be iid observations from uniform$(\theta - \frac{1}{2}, \theta + \frac{1}{2})$ distribution where $-\infty < \theta < \infty$ is the unknown parameter. Consider the square error loss for estimating θ, i.e., $L(\theta, d) = (d - \theta)^2$. Let $\hat{\theta}_n = \frac{X_{(1)} + X_{(n)}}{2}$, where $X_{(1)}$ and $X_{(n)}$ are the first and last order statistics, respectively.

1). Identify a minimal sufficient statistic for θ and show that it is indeed minimal.

2). Is the minimal sufficient statistic complete? Prove your answer.

3). Show that $\hat{\theta}_n$ is an maximum likelihood estimator of θ.

4). Is it an unbiased estimator? Justify your answer (no detailed calculation of the expression is needed).

5). Consider a prior distribution uniform $(-A, A)$ on θ with $A > 0$. Find a Bayes estimator of θ.

6). Prove that $\hat{\theta}_n$ is an equalizer estimator.