Ph.D. Qualifying Exam

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Instructions: Do any six of the nine complete questions. No materials.

1. Suppose that $f, g \in L^1(\mathbb{R}, m)$ where m denotes the Lebesgue measure and, for every $a < b$
$$\int_a^b f dm \geq \int_a^b g dm.$$
Show that m-almost everywhere, $f \geq g$.

2. Suppose that $g \in C([a,b])$ and $K \in C([a,b] \times [a,b])$. For each $u \in L^1([a,b], m)$ (m denotes Lebesgue measure on $[a,b]$) define
$$Tu(x) = g(x) + \int_a^b K(x,y)u(y)\, dm(y).$$
   (a) Show that $Tu \in C([a,b])$.
   (b) Show that $\{Tu : \|u\|_1 \leq 1\}$ is compact in $C([a,b])$. (Here $\|\cdot\|_1$ is the $L^1([a,b], m)$ norm.

3. (a) State the Stone Weierstrass Theorem.
   (b) Suppose that $f \in C(0, \pi)$ and $\int_0^\pi f(x) \cos nx\, dx = 0$ for all n. Show that $f(x) = 0$ for all x, $0 < x < \pi$. (Suggestion: The trigonometric identity $2 \cos A \cos B = \cos(A + B) + \cos(A - B)$ may be useful.)

4. Let $f_n(x) = n(\sin x)^n \cos x$.
   (a) Show that the sequence of functions f_n converges to 0 uniformly on any interval of the form $[0, a]$ where $a < \pi/2$.
   (b) Does f_n converge to 0 uniformly on $[0, \pi/2]$?
   (c) Show that, for any continuous function $g \in C([0, \pi/2])$
$$\lim_{n\to\infty} \int_0^{\pi/2} f_n(x)g(x)\, dx = g(\pi/2).$$

5. Prove or disprove each of the following three statements.

   (a) If $a_n \geq 0$ is a sequence of nonnegative real numbers and $\sum_n^\infty a_n$ exists then $\sum_n^\infty a_n^2$ exists.

   (b) If a_n is a sequence of real numbers and $\sum_n^\infty a_n$ exists then $\sum_n^\infty a_n^2$ exists.

   (c) If a_n is a sequence of real numbers and $\lim_{n\to\infty} a_n = 0$ and the partial sums $s_k = \sum_{n=1}^k a_n$ are uniformly bounded , then $\sum_{n=1}^\infty a_n$ exists.

6. (a) Define the sets of first and second categories in topological space and state Baire category theorem.

   (b) Show that the set of all transcendental numbers in the interval [0,1] is a set of second category in $[0, 1]$.[A number that is a solution of a polynomial equation with integer coefficients is called an algebraic number, for example $\sqrt{2}$. All others are called transcendental, for example e, π etc. are transcendental.

7. Define $f(x) = \dfrac{1 - \cos x}{x}$ for $x > 0$.

   (a) Show that $f \notin L^1(0, \infty)$.

   (b) Show that the improper Riemann integral R-$\displaystyle\int_1^\infty \frac{\cos x}{x}\,dx$ exists.

8. Define the convergence of the infinite product $\Pi_{n=1}^\infty(1 + a_n)$.

   (a) Prove that $\Pi_{n=1}^\infty \left(1 + \dfrac{1}{n}\right)$ diverges.

   (b) $\Pi_{n=1}^\infty \left(1 + \dfrac{1}{n^2}\right)$ converges.

9. Prove or disprove each of the following statements.

(a) If $f_n \in L^1(\mu), f_n \geq 0$ for $n \in \mathbb{N}$ and $\{f_n\}$ converges pointwise to $f \in L^1(\mu)$ as $n \to \infty$, then

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

(b) f is measurable if and only if $|f|$ is measurable.

(c) Let C be the middle-third Cantor set. Then the Lebesgue measure of C is zero and the characteristic function of C is Riemann integrable.