# Ph.D. QUALIFYING EXAM
# DIFFERENTIAL EQUATIONS
# Spring, 2009

This exam has two parts, ordinary differential equations and partial differential equations. In Part I, do problems 1 and 2 and choose two from the remaining problems. In Part II, Choose four problems.

## Part I: Ordinary Differential Equations

**1.** Consider the differential equation with initial condition

$$dx/dt = F(t,x), \quad x(a) = x_0 \in R^n$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ and
$F(t,x) = (F_1(t,x), F_2(t,x), \ldots, F_n(t,x))^T$. Suppose $F(t,x)$ is continuous for $a \leq t \leq b$ and $x \in R^n$ and satisfies a Lipschitz condition $|F(t,x) - F(t,y)| \leq L|x-y|$ for $a \leq t \leq b$ and all $x, y$.

(a) Convert the differential equation with the initial condition into an equivalent integral equation.

(b) Set up the Picard iteration process and prove that the sequence converges uniformly on the interval [a, b] to a limit function $x_\infty(t)$.

(c) Show that $x_\infty(t)$ is a solution to the differential equation on [a, b].

(d) Establish that the solution to the differential equation with the given initial condition is unique.

**2.** State and prove the Sturm Separation Theorem and Sturm Comparison Theorem for ODE's of the form $y'' + p(x)y = 0$.

**3.** Prove that any non-trivial solution to the Airy's equation $y'' + xy = 0$ has infinitely many zeros on $(0, \infty)$ but at most one zero on $(-\infty, 0)$. Find the two power series solutions satisfying the following initial conditions.

(a) $y(0) = 1, y'(0) = 0$;

(b) $y(0) = 0, y'(0) = 1$.

**4.** Consider the Sturm-Liouville system

$$y'' + \lambda y = 0, \quad y'(0) = 0, y(\pi) = y'(\pi).$$

(a) Find all eigenfunctions and eigenvalues $(y_n(x), \lambda_n)$. Be sure to check the possibilities $\lambda_n \leq 0$.

(b) Show that the set of eigenfunctions form an orthogonal set of functions in $L^2[0, \pi]$.

(c) Solve for the $\lambda_n$ graphically and make a good asymptotic estimate for $\lambda_n$.

**5.** Show that the autonomous system

$$\begin{aligned} \dot{x} &= x - y - x^3 - xy^2 \\ \dot{y} &= x + y - y^3 - x^2 y \end{aligned}$$

has a unique critical point that is unstable and a unique limit cycle. (Hint: convert to polar coordinates.)

**6.** Consider the nonlinear DE $\ddot{x} - x + x^3 = 0$.

(a) Find the integral curves and sketch the trajectories in the phase plane.

(b) Classify the critical points as vortex points or saddle points of the autonomous system defining these curves.

**Part II: Partial Differential Equations**

**1.** (Poisson's Formula on a Disk) Let $f(\theta)$ be a continuous and $2\pi$-periodic function with Fourier series

$$f(\theta) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\theta + b_k \sin k\theta).$$

Let

$$u(r,\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} r^k (a_k \cos k\theta + b_k \sin k\theta).$$

(a) Prove that the series for $u(r,\theta)$ converges uniformly on any disk $B_R = \{(r,\theta) \mid 0 \leq r \leq R\}$ with $R < 1$.

(b) Show how to rewrite the series for $u(r,\theta)$ in the form

$$u(r,\theta) = \int_0^{2\pi} f(\phi) P(r, \theta - \phi) d\phi$$

where $P$ is the Poisson kernel satisfying

$$P(r,\phi) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos\phi + r^2}.$$

(c) Prove that $\lim_{r \to 1^-} u(r,\theta) = f(\theta)$ uniformly.

**2.** Find all radially symmetric solutions of $\Delta u = 1$ in $R^n$. $(n \geq 2)$

**3.** Consider the eikonal equation $u_x^2 + u_y^2 = u^2$.

(a) Find all solutions of the form $u(x,y) = f(x)$.

(b) Use (a) to write down a general solution $u = u(x,y,a,b)$. (Hint: Use the fact that the PDE is invariant under rotations in the $xy$ plane.)

(c) Find the solution of the PDE satisfying the condition $u(x,x) = 2$.

**4.** Let $B^+ = \{(x,y) \mid x^2 + y^2 < 1, y > 0\}$ be the open half disk. Suppose $u(x,y) \in C^2(B^+) \cap C^0(\bar{B^+})$ satisfies $\Delta u = u_{xx} + u_{yy} = 0$ in $B^+$ and $u(x,0) = 0$. Prove that the extension $u(x,y)$ to $B$ defined as follows is a harmonic function on all $B$.

$$u(x,y) = \begin{cases} u(x,y) & \text{if } y \geq 0; \\ -u(x,-y) & \text{if } y < 0. \end{cases}$$

**5.** Solve the following initial value problem for Burger's equation.

$$\begin{aligned} u_t(x,t) + 2u(x,t)u_x(x,t) &= 0 \quad \text{on} \quad t > 0, \\ u(x,0) &= x. \end{aligned}$$

**6.** Consider the Fourier series solution to the heat equation with an initial condition and a boundary condition

$$\begin{cases} u_t = u_{xx}, & \text{for } 0 < x < \pi,\ t > 0, \\ u(x,0) = f(x), \\ u(0,t) = u(\pi,t) = 0. \end{cases}$$

(a) Derive the formal solution $u(x,t) = \sum_{k=1}^{\infty} b_k e^{-k^2 t} \sin kx$ where the $b_k$ are the coefficients of the Fourier series $\sum_{k=1}^{\infty} b_k \sin kx$ for the continuous function $f(x)$.

(b) Show that for every $\delta > 0$ this solution series converges uniformly in the region $0 \leq x \leq \pi, t \geq \delta$. Also show the same convergence for the corresponding series for $u_t, u_x, u_{xx}$.

(c) Show that $\lim_{t \to 0^+} u(x,t) = f(x)$ in the $L^2$ norm.