Ph.D. Real Analysis Qualifying Examination

January 24, 2009

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Instructions: Do six of the following questions. No materials are allowed.

1. (a) State the Baire Category Theorem. If you use the terminology "first category" or "second category" then you should define those terms.

   (b) Suppose that $\{f_\lambda : \lambda \in \Lambda\}$ is a collection of continuous complex valued functions defined on a complete metric space X. Suppose further that, for every $x \in X$, there is $\epsilon_x > 0$ so that $|f_\lambda(x)| > \epsilon_x$ for every $\lambda \in \Lambda$. Show that there is $\epsilon > 0$ and an open set $U \subseteq X$ so that $|f_\lambda(x)| > \epsilon$ for all $x \in U$ and $\lambda \in \Lambda$.

2. (a) Define equicontinuity.

   (b) State the Arzela Ascoli Theorem.

   (c) Suppose that f_n, $n \in \mathbb{N}$ is a sequence of differentiable functions defined on $[0, \infty)$ such that

$$\begin{aligned} \frac{d}{dx} f_n(x) &= (1 + x^2 + f_n(x)^4)^{-1/2} \quad \text{for} \quad x > 0 \\ f_n(0) &= \sin n \end{aligned}$$

   Show that, for any $b > 0$, there is a subsequence of the f_n which converges uniformly on $[0, b]$.

3. Suppose that μ is a Lebesgue Stieltjes measure (which means μ is a measure defined on the Borel subsets of $\mathbb{R}$ and is finite valued on bounded sets). Suppose that F is a corresponding distribution function which means that $\mu((a, b]) = F(b) - F(a)$ whenever $a < b$. Suppose further that F is continuously differentiable on some interval I with derivative F'. Show that for any bounded Borel measurable function g defined on I and any $a, b \in I$,

$$\int_{[a,b]} g d\mu = \int_{[a,b]} gF' dm$$

   where m is Lebesgue measure.

4. Suppose that f is a continuous function defined on a interval $I \subseteq \mathbb{R}$.

   (a) If $I = [0, 1]$ and $\epsilon > 0$ is given show that there are finitely many constants a_k, $1 \leq k \leq n$, so that

   $$|f(x) - \sum_{1 \leq k \leq n} a_k e^{-kx}| < \epsilon \quad \text{for all} \quad x \in I$$

   (b) Show that the statement in Part (a) is false if the interval I is $I = [0, \infty)$.

   (c) Suppose that f is continuous on $I = [0, \infty)$ and $\lim_{x \to \infty} f(x) = 0$. Is the conclusion in Part (a) true for such f?

5. (a) Show that the series $\sum_n \frac{1}{1+n^3x^2}$ converges uniformly for x in any compact subinterval of (0,1] but does not converge uniformly on (0,1] itself.

   (b) Define $f(x) = \sum_n \frac{1}{1+n^3x^2}$ for $0 < x \leq 1$. Show that f is not bounded.

   (c) Is $f \in L^1((0, 1])$?

6. (a) Show that a compact subset of a metric space is closed and bounded.

   (b) Is the converse true? Prove or give a counter example.

7. If f is real valued and satisfies the intermediate value property on $[ab]$ and further it is not continuous, then it must assume some value infinitely often. (Recall that f has the intermediate value property on $[a, b]$ if, whenever (c, d) is a subinterval of $[a, b]$, f assumes every value strictly between $f(c)$ and $f(d)$ on the interval (c, d).)

8. Show that any convex function defined on an open interval is differentiable almost everywhere.

9. (a) Define the convergence of an infinite product $\Pi_{n=1}^{\infty}(1 + u_n)$.

   (b) Show that it converges if the series $\displaystyle\sum_{n=1}^{\infty} |u_n|$ is convergent.