Topology Ph.D. Qualifying Exam

Gerard Thompson Mao-Pei Tsui

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This examination has been checked carefully for errors. If you find what you believe to be an error in a question, report this to the proctor. If the proctor's interpretation still seems unsatisfactory to you, you may modify the question so that in your view it is correctly stated, but not in such a way that it becomes trivial. If you feel that the examination is on the long side do not panic. The grading will be adjusted accordingly.

1 Part One: Do six questions

1. Prove that in a metric space a compact subset is closed and bounded. If you cannot do it for a general metric space do it for $\mathbb{R}^n$.

2. Prove that in a complete metric space (X, d) a subspace Y of X is complete if and only if it is a closed subspace of X.

3. Define the term *closure* $\overline{A}$ of a subset A of a topological space X. Prove that if B is a closed subset of X such that $A \subset B$ then $\overline{A} \subset B$.

4. Let (X, d) be a metric space and let $A \subset X$. If $x \in X$ define the distance of x to A to be $\inf\{d(x, a) : a \in A\}$. Prove that the real-valued function on X defined by $x \mapsto d(x, A)$ is continuous.

5. Prove that a closed subset of a compact topological space is compact. Prove that in a Hausdorff topological space a compact subset is closed.

6. Define the term *identification map* in the category of topological spaces. Let $\pi : X \to Y$ be a surjective, continuous map of topological spaces. Suppose that π maps closed sets to closed sets. Show that π is an identification map. What happens if we replace closed sets by open sets? Justify your answers.

7. Prove that the product topological space $X \times Y$ is Hausdorff if and only if X and Y are Hausdorff.

8. Prove that the open interval $(0, 1)$ considered as a subset of $\mathbb{R}$ in the usual topology is not compact.

9. Prove that $\mathbb{R}$ with the usual topology is connected.

10. Construct a topological space X by starting with $\mathbb{R}$ with the usual topology and defining x to be equivalent to y if $x - y$ is rational. Show that the resulting quotient or identification space X has the indiscrete topology, that is, the only open sets are $\emptyset$ and X.

11. A topological space X is said to be *locally connected* if the connected components of each point form a base of neighborhoods of X. Prove that in a locally connected space the connected components of X are open in X.

12. True or false: in a compact topological space every infinite set has a limit point. Justify your answer.

2 Part Two: Do three questions

1. Define what is meant by a Möbius band. Identify the space obtained by identifying the boundary of a Möbius band to a point. Give a brief explanation.

2. Prove that if $f, g : X \to S^{n-1}$ are continuous and both not surjective then f is homotopic to g.

3. Let X be a topological space and let $x_0 \in X$. Define the product of homotopy classes of loops $[\alpha]_{x_0}$ based at x_0 and verify in detail that this product is associative.

4. Give the definitions of deformation retract and strong deformation retract for topological spaces. Compute the fundamental group of $\mathbb{R}^3 - C$ where C denotes the circle $x^2 + y^2 = 1, z = 0$.

5. (i) The polygonal symbol of a certain surface without boundary is $xy^{-1}x^{-1}zwz^{-1}vyw^{-1}v^{-1}$. Identify the surface. What is its Euler characteristic?

   (ii) Explain how polygons with an even number of sides may be used to classify surfaces without boundary. You do not need to give detailed proofs.

6. It is known that if $p : \tilde{X} :\to X$ is a covering space and $x_0 \in X$ then the cardinality of $p^{-1}(x_0)$ is the index of $p_*\pi_1(\tilde{X}, y_0)$ in $\pi_1(X, x_0)$ where $p(y_0) = x_0$. Use this fact to deduce that there is no n-sheeted covering of the circle S^1 for any finite n.

7. Compute the Euler characteristic of the n-sphere S^n using the standard triangulation of an n-simplex.