Ph.D. Qualifying Exam in Real Analysis

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Instructions: Do 6 problems of 9. No materials are allowed. Complete explanations are expected.

1. (a) Define equicontinuity.

   (b) State the Arzelá Ascoli Theorem.

   (c) Let $\{a_n\}$, $n \in \mathbb{N}$ be a sequence of nonzero real numbers. Prove that the sequence of functions

   $$f_n(x) = \frac{1}{a_n}\sin(a_n x) + \cos(x + a_n)$$

   has a subsequence convergent to a continuous function.

2. Let $f \in L^1(\mathbb{R})$ and suppose that there is a countable set $S \subseteq \mathbb{R}$ so that

   $$\int_p^q f(x)\, dx = 0$$

   whenever p and q are *not* in S. Prove that $f = 0$ almost everywhere.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and suppose that x_n and y_n, $n = 1, 2, \ldots$ are two sequences such that $\lim_{n\to\infty} |x_n - y_n| = 0$. Does it follow that $\lim_{n\to\infty} |f(x_n) - f(y_n)| = 0$? Prove or give a counterexample.

4. Let $C(X)$ denote the continuous real valued functions defined on a compact set $X \subseteq \mathbb{R}$ and endowed with the sup norm topology.

   (a) Suppose that $T_0 \subseteq C(X)$ consists of all polynomials of the form $p(x) = a_0 + a_1x^2 + a_2x^4 + \ldots a_nx^{2n}$ for some real coefficients a_j, $0 \leq j \leq n$. Describe the closure of T_0 in $C(X)$ if $X = [0, 2]$

   (b) Describe the closure of T_0 in $C(X)$ if $X = [-2, 2]$

   (c) Suppose that $T_1 \subseteq C([0, 2])$ consists of all polynomials of the form $q(x) = a_0x + a_1x^3 + a_2x^5 + \ldots a_nx^{2n+1}$. Describe the closure of T_1 in $C([0, 2])$.

5. Let $f_n(x) = \dfrac{n \sin x}{x(1+n^2x^2)}$. Evaluate $\lim_{n\to\infty} \int_0^1 f_n \, dx$ or show that the limit does not exist.

6. Let $f : [1, \infty) \to [0, \infty)$ be a non-increasing function. Prove that

$$\int_1^\infty f(x)\, dx < \infty \quad \text{if and only if} \quad \sum_{k=0}^{\infty} 2^k f(2^k) < \infty.$$

7. Let $\mathcal{F}$ be a σ-algebra on a set Ω. For each $x \in \Omega$ define

$$A_x = \cap\{B : B \in \mathcal{F} \text{ and } x \in B\}.$$

(such a set is called an atom.) Prove that for all $x, y \in \Omega$, A_x and A_y are either identical or disjoint.

8. Let μ be a finite measure on a set Ω. Suppose f is a nonnegative measurable function defined on Ω such that f^n is integrable for all $n = 1, 2, \ldots$ and that

$$\int_\Omega f^n \, d\mu = \int_\Omega f \, d\mu$$

for all n. Show that $f = \chi_E$ a.e. for some measurable set $E \subseteq \Omega$. Is the result true if we do not assume that f is nonnegative?

9. Let $f : (0, \infty) \to \mathbb{R}$ be a convex function such that $\lim_{x\to 0} f(x) = 0$. Show that the function $x \mapsto \dfrac{f(x)}{x}$ is increasing on $(0, \infty)$.