Topology Ph.D. Qualifying Exam

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This examination has been checked carefully for errors. If you find what you believe to be an error in a question, report this to the proctor. If the proctor's interpretation still seems unsatisfactory to you, you may modify the question so that in your view it is correctly stated, but not in such a way that it becomes trivial. If you feel that the examination is on the long side do not panic. The grading will be adjusted accordingly.

1 Part One: Do six questions

1. Prove that in a compact metric space (X, d) there exists a constant K such that for all $x, y \in X$, $d(x, y) \leq K$.
2. For any subset A of a topological space show that $\overline{A} - A$ has empty interior.
3. Prove that the intersection of two open dense subsets of a topological space is dense.
4. Let (X, d) be a metric space and let $A \subset X$. If $x \in X$ define the distance of x to A to be $\inf\{d(x, a) : a \in A\}$. Prove that the real-valued function on X defined by $x \mapsto d(x, A)$ is continuous.
5. Let X be a topological space. The diagonal of $X \times X$ is the subset $\triangle = \{(x, x) : x \in X\} \subset X \times X$. Show that X is Hausdorff if and only if $\triangle$ is closed in $X \times X$.
6. Prove the equivalence of the following:

   (i) $\tilde{X} := X/\sim$ has the quotient or identification topology

   (ii) $\tilde{X}$ has the strongest topology to make the quotient map $\pi : X \to \tilde{X}$ continuous.

   (iii) For all topological spaces Y and all maps $f : X \to Y$ the map f is continuous if and only if the map $\tilde{f} : \tilde{X} \to Y$ such that $f = \tilde{f} \circ \pi$ is continuous.
7. Define compactness for a topological space. A collection of subsets is said to have the finite intersection property if every finite class of those subsets has non-empty intersection. Prove that a topological space is compact iff every collection of closed subsets that has the finite intersection property itself has non-empty intersection.
8. A topological space is said to be paracompact if it Hausdorff and every open cover has a locally finite refinement. Prove that a Hausdorff, second countable space is paracompact.

9. Let $f : X \mapsto Y$ be a quotient map, with Y connected. Show that if $f^{-1}(y)$ is connected for all $y \in Y$, then X is connected.

10. Let X be a discrete space, Y be a space with the trivial topology, and Z be any topological space. Show that any maps $f : X \mapsto Z$ and $g : Z \mapsto Y$ are continuous. If Z is Hausdorff, show that the only continuous maps $h : Y \mapsto Z$ are constant maps.

11. Prove that a topological space X is disconnected iff there is a non-constant continuous map from X to $\{0, 1\}$ where the latter space is given the discrete topology, that is, every subset is open.

12. A topological space X is T_1 if given any two distinct points each has a neighborhood that is disjoint from the other. Prove that X is T_1 if and only its points are closed.

2 Part Two: Do three questions

1. Let $X = [0, 1] \times [0, 1]$ denote the rectangle in R^2. Let $\sim$ be the equivalent relation generated by $(0, p) \sim (1, 1 - p)$ where $0 \leq p \leq 1$. The quotient space $X/\sim$ is called the Möbius band. Show that S^1 is a retract of the Möbius band.

2. Recall that $S^k = \left\{(x_1, \cdots, x_{k+1}) \middle| x_1^2 + \cdots x_{k+1}^2 = 1\right\} \subset R^{k+1}$. The antipodal map $A_k : S^k \mapsto S^k$ is the smooth map defined by $(x_1, \cdots, x_{k+1}) \mapsto (-x_1, \cdots, -x_{k+1})$.

   (a) Show that $A_1 : S^1 \mapsto S^1$ is homotopic to the identity map.

   (b) Show that $A_k : S^k \mapsto S^k$ is homotopic to the identity map if k is odd.

3. (a) The space G is a topological group meaning that G is a group and also a Hausdorff topological space such that the multiplication and map taking each element to its inverse are continuous operations. Given two loops based at the identity e in G, say $\alpha(s)$ and $\beta(s)$, we have two ways to combine them: $\alpha \cdot \beta$ (product of loops as in the definition of fundamental group) and secondly $\alpha\beta$ using the group multiplication. Show, however, that these constructions give homotopic loops.

   (b) Show that the fundamental group of a path-connected topological group is abelian. (Hint: Show that $\alpha\beta \sim \beta\alpha$.)

4. Give the definitions of deformation retract and strong deformation retract for topological spaces. Compute the fundamental group of $\mathbb{R}^3 - C$ where C denotes the circle $x^2 + y^2 = 1, z = 0$.

5. The polygonal symbol of a certain surface without boundary is $ab^{-1}a^{-1}bcc$. Identify the surface. What is its Euler characteristic?

6. Let X be the quotient space of the sphere S^2 where we identified the north pole and the south pole. Compute the fundamental group of X.

7. Let $n \geq 2$. Compute the Euler characteristic of the n-sphere S^n using the standard triangulation of an $n + 1$-simplex. (Hint: The proper faces of a $n + 1$ simplex is homeomorphic to S^n.)