ALGEBRA PH.D. QUALIFYING EXAM, JANUARY 22, 2011

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Instructions: Please do *six* problems, including *at least one problem from each of the five sections.* Give complete proofs. If you attempt more than six problems, indicate clearly which six problems you would like to be graded. You have three hours to complete the exam.

$\mathbb{C}$ = the field of complex numbers
$\mathbb{Q}$ = the field of rational numbers.

1. Linear Algebra

1. Let A be the 2×2 integral matrix $\begin{pmatrix} 1 & 1 \\ -4 & 5 \end{pmatrix}$. Determine for which positive integers n, there is a complex matrix B such that $B^n = A$. Prove your assertions!

2. Let F be an algebraically closed field and let $M_n(F)$ be the ring of $n \times n$ matrices over F . Describe those matrices $X \in M_n(F)$ with the property that all matrices that commute with X are diagonalizable.

2. Groups

3. Let $\mathbb{Z}_2$ be the group of order 2, D_m be the dihedral group of order $2m$ and let n be an *odd* number.

(a) Prove that the groups D_{2n} and $D_n \times \mathbb{Z}_2$ are isomorphic.

(b) Prove that the groups D_{4n} and $D_{2n} \times \mathbb{Z}_2$ are not isomorphic.

4. Let $G \neq 1$ be a (possibly infinite) group whose subgroups are linearly ordered by inclusion. In other words, if H and K are subgroups of G, then either $H \leq K$ or $K \leq H$.

(a) Prove that G is an abelian group and that for some prime p, the orders of the elements of G are all powers of p.

(b) If $G_n = \{g \in G \mid g^{p^n} = 1\}$, prove that $|G_n| \leq p^n$.

3. Fields

5. Let $F = \mathbb{Z}_3$, the field of 3 elements, and let $f = X^3 - X + 1 \in F[X]$.

(a) Give an explicit construction for a splitting field for f over F.

(b) Determine the degree $|E : F|$ and the number of elements in E.

(c) Find an explicit factorization for f over E.

6. (a) Find the minimal polynomial of $\sqrt{2} + \sqrt[3]{5}$ over $\mathbb{Q}$.

(b) If f denotes this minimal polynomial and E is a splitting field of f over $\mathbb{Q}$, what is the degree $|E : \mathbb{Q}|$? Prove your assertions!

4. Galois Theory

In each of the following two problems, given a field extension L/M, $G(L/M)$ denotes the group of all automorphisms of L that fix each element of M.

7. If E/F is a finite Galois extension and K is an intermediate field, prove that $G(K/F) \cong N_G(H)/H$ (where $G = G(E/F)$, $H = G(E/K)$ and $N_G(H)$ is the normalizer of H in G). Is the extension K/F necessarily Galois? Justify your answer.

8. Let p be an odd prime and let $E = \mathbb{Q}(\zeta + \zeta^{-1})$, where $\mathbb{Q}$ is the field of rational numbers and $\zeta \in \mathbb{C}$ is a primitive pth root of unity. Determine the group $G(E/\mathbb{Q})$. Is the extension $E/\mathbb{Q}$ Galois? Justify your answer.

5. Rings, Modules, Algebras

9. Let F be a commutative field, $R = F[X]$, $S = F[Y]$, and $T = F[X, Y]$. Prove that T is the coproduct of R and S in the category of all commutative F-algebras.

10. Let R be a commutative ring and let X be a non-empty subset of R such that $0 \notin X$ and X is closed under multiplication.

(a) Prove that there exists an ideal A of R such that $A \cap X = \emptyset$ but $I \cap X \neq \emptyset$ if I is an ideal properly containing A.

(b) Prove that an ideal A satisfying the property described in part (a) is necessarily prime.