1. Let $\mathcal{F}$ denote the space of all distribution functions and $T = T(\cdot)$ be a statistical functional on $\mathcal{F}$. Given two points F and G in the space $\mathcal{F}$, suppose that T has a differential $\phi'_F(G-F)$ at F with respect to a norm $||\cdot||$. Show that for any G, $T'_F(G-F)$ exists and

$$T'_F(G-F) = \phi'_F(G-F),$$

where $T'_F(G-F)$ is the Gâteaux differential of T at F in the direction of G.

2. Let $T(F)$ denote the variance functional

$$T(F) = \int x^2 dF(x) - \left(\int x dF(x)\right)^2.$$

(a) Find the Gâteaux differential $T'_F(G-F)$ of T at F in the direction of G.

(b) Find the second-order Gâteaux differential $T''_F(G-F)$ of T at F in the direction of G.

(c) Suppose $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} F$. Let $F_n(x) = n^{-1}\sum_{i=1}^n I(X_i \leq x)$ be the empirical distribution function. Find an expression for $R_{1n} = T(F_n) - T(F) - T'_F(F_n - F)$ for each n and show that $\sqrt{n}R_{1n} \xrightarrow{p} 0$ as $n \to \infty$.

3. Suppose $X_1, \cdots, X_n$ are iid Bernoulli random variables with $p = P[X_1 = 1], 0 < p < 1$. Show that

$$T = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}$$

is the UMVUE of $p(1-p)$.

4. Suppose $\{X_n, n \geq 1\}$ are independent r.v.'s such that

$$P[X_n = 1] = P[X_n = -1] = \frac{1}{2n} \text{ and } P[X_n = 0] = 1 - \frac{1}{n}.$$

Determine whether the CLT holds by determining whether the Lindeberg condition holds.