## Department of Mathematics The University of Toledo

## Ph.D. Qualifying Examination Probability and Statistical Theory

January 29, 2011

Instructions

Do all four problems.

Show all of your computations. Prove all of your assertions or quote appropriate theorems. This is a closed book examination. This is a three hour test. 1. Let  $\mathcal{F}$  denote the space of all distribution functions and  $T = T(\cdot)$  be a statistical functional on  $\mathcal{F}$ . Given two points F and G in the space  $\mathcal{F}$ , suppose that T has a differential  $\phi'_F(G-F)$  at F with respect to a norm  $||\cdot||$ . Show that for any G,  $T'_F(G-F)$  exists and

$$T'_F(G-F) = \phi'_F(G-F),$$

where  $T'_F(G-F)$  is the Gâteaux differential of T at F in the direction of G.

**2.** Let T(F) denote the variance functional

$$T(F) = \int x^2 dF(x) - \left(\int x dF(x)\right)^2.$$

- (a) Find the Gâteaux differential  $T'_F(G-F)$  of T at F in the direction of G.
- (b) Find the second-order Gâteaux differential  $T''_F(G-F)$  of T at F in the direction of G.
- (c) Suppose  $X_1, \ldots, X_n \stackrel{\text{i.i.d.}}{\sim} F$ . Let  $F_n(x) = n^{-1} \sum_{i=1}^n I(X_i \leq x)$  be the empirical distribution function. Find an expression for  $R_{1n} = T(F_n) T(F) T'_F(F_n F)$  for each n and show that  $\sqrt{nR_{1n}} \xrightarrow{p} 0$  as  $n \to \infty$ .

3. Suppose  $X_1, \dots, X_n$  are iid Bernoulli random variables with  $p = P[X_1 = 1], 0 .$  $Show that <math>\sum_{n=1}^{n} (X_n = \bar{X})^2$ 

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$$T = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n - 1}$$

is the UMVUE of p(1-p).

4. Suppose  $\{X_n, n \ge 1\}$  are independent r.v.'s such that

$$P[X_n = 1] = P[X_n = -1] = \frac{1}{2n}$$
 and  $P[X_n = 0] = 1 - \frac{1}{n}$ .

Determine whether the CLT holds by determining whether the Lindeberg condition holds.