Ph.D. Qualifying Examination

Real Analysis

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Instructions: Do 6 of the 8 problems. If you do more, then state which should be graded.

1. Suppose that $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Show that $f \in L^p(\mathbb{R})$ for all $p \geq 1$ and
$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$$
where $\|\cdot\|_p$ denotes the $L^p(\mathbb{R})$ norm.

2. Let (X, d) be a compact metric space. Suppose that $f : X \to X$ is an isometry (which means that $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$). Show that f is surjective.

3. (a) State the Arzela-Ascoli Theorem.

   (b) Suppose that $\{f_n \colon n \in \mathbb{N}\}$ is a sequence of functions in $C^1([0,1])$ (so that f_n is continuously differentiable on $[0, 1]$). Suppose further that, for all $n \in \mathbb{N}$
$$|f_n'(x)| \leq \frac{1}{\sqrt{x}}, \text{ for all } x,\ 0 < x \leq 1$$
and
$$\int_0^1 f_n(x)\,dx = 0.$$
Show that a subsequence of the sequence f_n must converge uniformly on [0,1].

4. Consider the sequence of functions $f_n(x) = \dfrac{ne^x}{1+n^2x^2}$, $n \in \mathbb{N}$, defined on $[0, 1]$. Evaluate the limit: $\lim_{n\to\infty} \int_0^1 f_n(x)dx$.

5. (a) Let $1 \leq p < \infty$. Show that, if a sequence of real-valued functions $\{f_n\}$, $n \geq 1$ converges in L^p, then a subsequence converges almost everywhere.

(b) Give an example of a sequence of functions converging to zero in $L^2(\mathbb{R})$ that does not converge almost everywhere.

6. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable, $f(0) = 0$, and $f'(x) > f(x)$ for all $x \in \mathbb{R}$. Prove that $f(x) > 0$ for $x > 0$.

7. Let $p > 1$ and define $\phi : \mathbb{R} \to \mathbb{R}$, by

$$\phi(x) = \begin{cases} x^{-1/p} & \text{if } 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Let $\{r_n : n \in \mathbb{N}\}$ be a countable dense subset of $\mathbb{R}$ and define

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} \phi(x - r_n)$$

(a) Prove that $f \in L^1(\mathbb{R})$ so that, in particular, $f(x) < \infty$ for almost all x.

(b) Prove that f^p is not integrable on any interval (a, b), $a < b$.

8. (a) State the Baire Category Theorem.

(b) Suppose $\{f_n\}$ is a sequence of real valued continuous functions defined on a complete metric space X and converging pointwise to 0. Show that given any $\epsilon > 0$, there exists a non-empty open set $U \subseteq X$ and a positive integer N such that $|f_n(x)| < \epsilon$ for all x in U and all $n > N$.