# Topology Ph.D. Qualifying Exam

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This examination has been checked carefully for errors. If you find what you believe to be an error in a question, report this to the proctor. If the proctor's interpretation still seems unsatisfactory to you, you may modify the question so that in your view it is correctly stated, but not in such a way that it becomes trivial. If you feel that the examination is on the long side do not panic. The grading will be adjusted accordingly.

## 1 Part One: Do six questions

1. Prove that in a compact metric space $(X, d)$ there exists a constant $K$ such that for all $x, y \in X$, $d(x, y) \leq K$.

2. Define what it means for $(X, d)$ to be a metric space. Then $d : X \times X :\to \mathbb{R}$: is $d$ continuous? Discuss.

3. Define compactness for a topological space. Prove from your definition that the closed interval $[0, 1]$ is compact. (Here $[0, 1] \subset \mathbb{R}$ with the standard topology).

4. Let $X = \Pi_{\mu \in M} X_\mu$ and $Y = \Pi_{\mu \in M} Y_\mu$ be the Cartesian products of the topological spaces $(X_\mu)_{\mu \in M}$ and $(Y_\mu)_{\mu \in M}$ and let $X$ and $Y$ have the product topologies, respectively. Prove that if for each $\mu \in M$ the maps $f_\mu : X_\mu \to Y_\mu$ are continuous then $f : X \to Y$ defined by $f(x)_\mu = f_\mu(x_\mu)$ is continuous.

5. Let $X, Y$ be topological spaces. Suppose that $X$ is compact and Y is Hausdorff. Let $f : X \mapsto Y$ be continuous and bijective. Prove that $f$ is a homeomorphism.

6. A topological space $X$ is said to be *regular* if a set that contains just one point and a closed set that are disjoint can be separated by disjoint open sets. Prove that in a regular space a closed set and a compact set that are disjoint can be separated by disjoint open sets.

7. Suppose $A = \cup_\alpha A_\alpha$, where each $A_\alpha$ is connected, and so that there is a point x common to all $A_\alpha$. Prove that $A$ is connected.

8. Let $A$ be a connected subset of a connected space $X$, and $B \subset X \backslash A$ be an open and closed set in the topology of the subspace $X \backslash A$ of the space $X$. Prove that $A \cup B$ is connected.

9. Let $f : X \mapsto Y$ be a continuous map between topological spaces $X$ and $Y$.
   (a) Define what it means for $f$ to be a quotient (an identification) map.
   (b) Prove that if the map $f : X \mapsto Y$ is open and onto, then $f$ is a quotient map.

10. Prove that a pathwise connected topological space is connected.

11. Recall that $g : X \mapsto Y$ is called a proper map if $g^{-1}(C)$ is compact whenever $C \subset Y$ is compact. Show that if a map $f : X \mapsto Y$ is closed and $f^{-1}(y)$ is compact for all $y \in Y$, then $f$ is proper.

12. Let $X$ and $Y$ be topological spaces with $Y$ is Hausdorff and let $f, g : X \mapsto Y$ be continuous functions. Prove that the set $\{x \in X | f(x) = g(x)\}$ is closed in $X$.

## 2 Part Two: Do three questions

1. Let $\mathbb{D}^n$ be the $n$-dimensional ball and $S^{n-1}$ the $n-1$dimensional sphere, realized as the boundary of $\mathbb{D}^n$. Prove that the following are equivalent:
   1. There is no retraction $\mathbb{D}^n \mapsto S^{n-1}$.
   2. Every continuous map $\mathbb{D}^n \mapsto \mathbb{D}^n$ has a fixed point.

2. Compute the fundamental group of the open subset of $\mathbb{R}^3$ obtained by removing the $x$ axis and the $y$ axis.

3. Suppose $f : S^2 \mapsto \mathbb{R}$ is a continuous map. Prove the there is a point $x \in S^2$ such that $f(x) = f(-x)$.

4. The polygonal symbol of a certain surface without boundary is $acdb^{-1}a^{-1}cbd^{-1}$. Identify the surface. What is its Euler characteristic?

5. Let $X$ be the union of the (surface of the) unit sphere in $\mathbb{R}^3$ with the unit disk in the $xy$ plane. Compute $\pi_1(X)$.

6. Give the definition of a covering space. Let $p : \widetilde{X} \mapsto X$ be a covering map and assume that $X$ is connected. Prove that the cardinality of the fibers $p^{-1}(x)$ where $x \in X$ is the same for all fibers.

7. Show that $\pi_1(X \times Y)$ is isomorphic to $\pi_1(X) \times \pi_1(Y)$ where $X$ and $Y$ are connected topological spaces.