Ph.D. QUALIFYING EXAM
DIFFERENTIAL EQUATIONS
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This exam has two parts, ordinary differential equations and partial differential equations. Choose four problems in each part.

Part I: Ordinary Differential Equations

1. Consider the sequence of functions $\{y_k(t)\}$ defined by $y_0(t) = 2 - 3t$ and

$$y_{k+1}(t) = 2 - \int_0^t (3 + \cos(y_k(s)))ds, \quad k = 0, 1, \ldots.$$

Prove that on any finite interval the sequence of the functions converges uniformly.

2. Let M be a manifold and $\phi(t, \cdot) : M \to M$ be a family of diffeomorphisms with $\phi(0, \cdot)$ equal to identity. ($(M, \phi(t, \cdot))$ is a dynamical system.) A set $S \subset M$ is called **minimal** if 1) it is nonempty, closed, and invariant of the diffeomorphisms; and 2) it does not contain any such set as a proper subset. Prove the following result:

If S is compact, then S is minimal if and only if for each $x \in S$, we have $S = \omega(x)$, where $\omega(x)$ is the ω-limit set of the curve $\phi(t, x)$. (Recall that the ω-limit set for $\phi(t, x)$ is the set of all $y \in M$ for which there exists an increasing sequence of times $\{t_k\}_{k\in\mathbb{N}}$, with $\lim_{k\to+\infty} t_k = +\infty$ such that $\lim_{k\to+\infty} \phi(t_k, x) = y$.)

3. Consider the following initial value problem:

$$\dot{x} = \begin{cases} x\ln(|x|) & x \neq 0 \\ 0 & x = 0 \end{cases} \qquad x(0) = x_0$$

where $x \in \mathbb{R}$.

(a) Explain if there is a solution with $x_0 = 0$.

(b) If a solution with $x_0 = 0$ exists, is it unique? Prove it or provide a counterexample.

4. Compute eigenvalues and eigenfunctions of the Sturm-Liouville problem.(Note that the eigenvalues are expressed in terms of the roots of a transcendental equation that can not be solved exactly, so estimate graphically the position of these roots.)

$$y''(x) + \lambda y(x) = 0,\ 0 < x < \pi,$$

$$y'(0) - y(0) = 0,\ y(\pi) = 0.$$

5. Solve the nonhomogeneous linear system for $x \in R^2$ with the initial condition.

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} x + \begin{bmatrix} e^{2t} \\ 1 \end{bmatrix} \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

6. Consider the nonlinear DE $\ddot{x} + (x^2 - 1)\dot{x} + x = 0$.

(a) Determine all the equilibrium points in the phase plane.

(b) Determine the stability of these equilibria by finding a suitable Lyapunov function.

7. Let Σ be the space of double sequences $\{\sigma_k\}_{k\in\mathbb{Z}}$ with values in the alphabet $\{0, 1\}$. Equip Σ with the metric

$$\delta(\sigma, \sigma') = \sum_{k\in\mathbb{Z}} 2^{-|k|}\delta(\sigma_k, \sigma'_k), \quad \delta(\sigma_k, \sigma'_k) = \begin{cases} 0 & \sigma_k = \sigma'_k \\ \\ 1 & \sigma_k \neq \sigma'_k \end{cases}.$$

Equip (Σ, δ) with the dynamic $\Psi : \Sigma \to \Sigma$ such that $(\Psi(\sigma))_k = \sigma_{k+1}$, so that (Σ, δ, Ψ) is a discrete dynamical system.

(a) Prove that the periodic points for Ψ are dense in (Σ, δ).

(b) Prove that the dynamic is transitive, namely that there exist orbits that are dense in (Σ, δ).

Part II: Partial Differential Equations

1. Find the solution to the heat equation with the initial-boundary values:

$$\begin{cases} u_t = u_{xx}, \quad \text{for } \ 0 < x < \pi, \ t > 0, \\ u(x,0) = \frac{1}{3}\sin(\pi x) - 2\sin(5\pi x), \quad \text{for } \ 0 < x < \pi, \\ u(0,t) = u(\pi,t) = 0, \quad \text{for } \ t > 0. \end{cases}$$

2. Find all radially symmetric solutions of $\Delta u - u = |x|^2$ in R^3.

3. Let us consider the following boundary value problem for the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad (0 < x < l_1, \ 0 < t < l_2),$$

$$u(0,t) = 0, \ u(l_1,t) = 0 \quad (0 < t < l_2)$$

$$u(x,0) = 0, \ u(x,l_2) = 0 \quad (0 < x < l_1).$$

Assuming l_2/l_1 is a rational number. Determine if the solution to the problem is unique or not in the given rectangle.

4. Let B^+ be the open half disk $\{(x,y) \mid x^2+y^2 < 1, y > 0\}$. Suppose $u(x,y) \in C^2(B^+) \cap C^1(\bar{B^+})$ satisfies $\Delta u = u_{xx} + u_{yy} = 0$ in B^+ and $u_y(x,0) = 0$. Prove that the extension $u(x,y)$ to B defined as follows is a harmonic function on all B.

$$u(x,y) = \begin{cases} u(x,y) & \text{if } y \geq 0; \\ u(x,-y) & \text{if } y < 0. \end{cases}$$

5. Let Q be an open connected bounded domain and let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of continuous functions on the closure $\bar{Q}$. Suppose the functions are harmonic on Q and the sequence converges uniformly on ∂Q.

(a) Prove that the sequence converges uniformly on all $\bar{Q}$.

(b) Prove that the limit function is a C^2 harmonic function on Q.

6. Consider the following initial value problem

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}, \quad (t > 0),$$

$$u(x,0) = f(x), \quad \frac{\partial u(x,0)}{\partial t} = g(x), \quad (-\infty < x < \infty).$$

Suppose $f(x) = g(x) = 0$ for $x \leq 0$. Show that $u(x,t) = 0$ if $x \leq -2t$ for $t > 0$.

7. Consider the initial-boundary value problem:

$$u_t(x,t) + u_x(x,t) = x, \quad x > 0,\ t > 0,$$

$$u(0,t) = t,\ t > 0,$$
$$u(x,0) = \sin(x),\ x > 0.$$

Solve this problem using the method of characteristics. Is the solution continuous along the line $x = t$? What happens instead if we choose as initial value $u(x,0) = \cos(x)$?