Department of Mathematics and Statistics
The University of Toledo

Ph. D. Qualifying Examination
Theory of Statistics

January 21, 2012

Instructions:

Do all four problems;

Show all of your computations;

Prove all of your assertions or quote appropriate theorems;

This is three-hour closed book examination.

1. Let $X = (X_1, \cdots, X_n) \overset{iid}{\sim} N(\mu, \sigma^2)$, where both $\mu \in \mathbb{R}$ and $\sigma^2 > 0$ are unknown parameters. Consider estimating μ^2.

   a. Show that $T_n = \bar{X}_n^2 - \frac{\sum(X_i - \bar{X}_n)^2}{n(n-1)}$ is the UMVUE;

   b. Find the Cramér-Rao Lower Bound;

   c. Find the limiting distributions of $c_n(T_n - \mu^2)$ by choosing an appropriate sequence of real numbers c_n which satisfies $c_n \to \infty$.

2. Let $X = (X_1, ..., X_n) \sim_{iid} E(a, \theta)$ with $a \in \mathbb{R}$ and $\theta > 0$.

   a. Find the UMVUE of a when θ is known.

   b. Find the UMVUE of θ when a is known.

   c. Assume that θ is known. Find the UMVUE of $P[X_1 \geq t]$ and $\frac{d}{dt}P[X_1 \geq t]$ for a fixed $t > 0$.

Let (X,Y) be distributed over the square with vertices at [0,0], [β,0], [0,β] and [β,β], with density proportional to the product xy, and with parameter space $\beta \in (0,\infty)$. That is, on this square, the joint density of (X,Y) is f(x,y) = kxy.

a. Find k.
b. Write and sketch the likelihood function as a function of β based upon the observed pair (x,y).
c. Based on one observation of the vector (X,Y), show that M=max(X,Y) is a sufficient statistic for β.
d. Find the CDF of M.
e. Based upon this random sample of size 1 (of the vector (X,Y)), find the likelihood ratio test for testing H_0:β=1 versus the alternative H_1:β ≠ 1 with level of significance α; i.e., find a test statistic and the exact critical region as a function of α.

Now assume α = .05.

f. If the observation is (.4,.9), find the P-value. What is your conclusion?
g. If the observation is (.4,.4), find the P-value. What is your conclusion?
h. Find and plot the power function for this test.
i. Now assume that we have n i.i.d. pairs of observations, $(X_1,Y_1),\ldots,(X_n,Y_n)$, from this distribution. Note that each pair yields $M_i = \max(X_i,Y_i)$,. Find the Method Moments estimator for β based on this random sample of $M_1, \ldots M_n$.
j. Find the mean square error of the method of moments estimator from part i. You may use the fact that $Var(M_1) = 2\beta^2/75$ without spending the time to derive it.
k. For testing H_0:β=1 versus the alternative H_1:β ≠ 1, find a test statistic based upon $M_1, \ldots M_n$ that, for large n and under H_0, has an approximate standard normal distribution.

4. Let $X_1, \ldots, X_n$ be a random sample drawn from a population with density $p(x;\theta)$, where θ is a p-dimensional parameter contained in a parameter space Θ. Write $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{x} = (x_1, \ldots, x_n)$, where $\mathbf{x}$ is the observed value of $\mathbf{X}$. Let $\mathcal{X}^n$ denote the sample space of $\mathbf{X}$ and $L(\theta;\mathbf{x}) = \prod_{i=1}^n p(x_i;\theta)$ represent the likelihood function of θ based on the data $\mathbf{x}$. For a statistical model $\mathcal{P} = \{p(x;\theta) : \theta \in \Theta \subset \mathcal{R}^p\}$, show that a statistic $T(\mathbf{X})$ is sufficient for θ if for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}^n$, the equality $T(\mathbf{x}) = T(\mathbf{y})$ implies that $L(\theta;\mathbf{y}) = m(\mathbf{x},\mathbf{y})L(\theta;\mathbf{x})$ for all $\theta \in \Theta$, where $m(\mathbf{x},\mathbf{y})$ is some function of $(\mathbf{x},\mathbf{y})$ independent of θ.