Ph.D. Qualifying Examination
Real Analysis
January 28, 2012.

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Instructions: Do 6 problems of 8. If you attempt more than 6 then indicate clearly which six are to be considered. No materials are allowed.

1. Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous, and for all $x \neq 0$, $f'(x)$ exists. If $\lim_{x\to 0} f'(x) = L$ exists, then does it follow that $f'(0)$ exists? Prove or disprove.

2. Let g be a continuous, real valued function defined on [0,1]. Determine, with proof, conditions on g which are equivalent to the property that $\lim_{n\to\infty} \|g^n f\|_2 = 0$ for all $f \in L^2(0,1)$.

3. Let a measurable bounded set $X \subseteq \mathbb{R}^n$ have the property that every continuous map $f : X \to \mathbb{R}$ is uniformly continuous. Show that X is compact.

4. Suppose that f and g are positive, continuous functions defined on $[a, \infty)$ for some $a \in \mathbb{R}$ and suppose $\int_a^\infty g(x)\,dx$ diverges (as an improper Riemann integral). Show that at least one of the integrals

$$\int_a^\infty f(x)g(x)\,dx, \qquad \int_a^\infty \frac{g(x)}{f(x)}\,dx$$

diverges.

5. Suppose that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous bijection of $\mathbb{R}$ onto $\mathbb{R}$ then $f(A)$ is a Borel set whenever A is a Borel set.

6. For $f \in L^1([0, e])$ evaluate the limit (if it exists)

$$\lim_{n\to\infty} \int_0^e n \log(1 + x/n) f(x)\,dx.$$

7. (a) State the Stone Weierstrass Theorem in the case of complex valued functions.

   (b) Recall that $\cosh x = (e^x + e^{-x})/2$. Suppose that, for some continuous function $f : \mathbb{R} \to \mathbb{R}$

$$\int_0^1 f(x)(\cosh x)^{2n}\, dx = 0$$

   for all $n \geq 1$. Show that $f(x) = 0$, for all x, $0 \leq x \leq 1$.

   (c) Give an example of a continuous function f so that

$$\int_{-1}^1 f(x)(\cosh x)^{2n}\, dx = 0$$

   for all $n \geq 1$, but $f(x)$ is not identically zero for $-1 \leq x \leq 1$.

8. Let $\{f_n\}$ be a sequence of nonnegative measurable functions converging pointwise to some measurable function f on $\mathbb{R}$ and suppose that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n\, dx = \int_{\mathbb{R}} f\, dx < \infty.$$

   Show that for each measurable set A,

$$\lim_{n\to\infty} \int_A f_n\, dx = \int_A f\, dx.$$