Topology Ph.D. Qualifying Exam

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This examination is scheduled to last two hours. Do not do more than six problems. The examination has been checked carefully for errors. If you find what you believe to be an error in a question, report it to the proctor. If the proctor's interpretation still seems unsatisfactory to you, you may modify the question so that in your view it is correctly stated, but not in such a way that it becomes trivial. If you feel that the examination is on the long side do not panic. The grading will be adjusted accordingly. A passing grade will be obtained for *complete* answers to four questions.

1 Part One: Do six questions

Recall that the k-sphere is defined to be $\mathbb{S}^k = \left\{(x_1, \cdots, x_{k+1}) \in \mathbb{R}^{k+1} \middle| x_1^2 + \cdots x_{k+1}^2 = 1\right\}$.

1. Define $f : \mathbb{S}^1 \to \mathbb{S}^1$ by $f(x) = -x$. Prove that f is homotopic to the identity map. Now suppose that $g : \mathbb{S}^1 \to \mathbb{S}^1$ is a map that is not homotopic to the identity map. Show that $g(x) = -x$ for some $x \in \mathbb{S}^1$.

2. (i) State the simplicial approximation theorem.
   (ii) Use the simplicial approximation theorem to compute the fundamental group of $\mathbb{S}^n$ for $n > 1$. You may assume that for $n = 1$ the fundamental group is $\mathbb{Z}$.

3. (i) Explain how polygons with an even number of sides may be used to classify surfaces without boundary. You do not need to give detailed proofs.

   (ii) The polygonal symbol of a certain surface without boundary is $xyzyzx$. Identify the surface. What is its Euler characteristic? Is the surface orientable?

4. (i) Prove that a retraction map (or retract) induces an epimorphism of fundamental groups. All that you have to prove here is the surjectivity.

   (ii) Let $X = [0, 1] \times [0, 1]$ denote the unit square in $\mathbb{R}^2$. Let $\sim$ be the equivalence relation generated by $(0, p) \sim (1, 1 - p)$ where $0 \leq p \leq 1$. The quotient space $X/\!\sim$ is called the Möbius band. Show that $\mathbb{S}^1$ is a retract of the Möbius band and hence compute the fundamental group of the Möbius band.

5. (i) State the Seifert-van Kampen Theorem for a topological space that is homeomorphic to the polyhedron of a simplicial complex.

(ii) For the sake of this problem a manifold of dimension n will be defined as a Hausdorff topological space in which each point has a neighborhood that is homeomorphic to $\mathbb{R}^n$. If M is a connected manifold of dimension at least 3 and $q \in M$, show that $\pi_1(M - \{q\})$ is isomorphic $\pi_1(M)$.

6. Let $\mathbb{S}^\infty \subset l^2(\mathbb{R})$ be the space of sequences which are square summable and of norm 1, that is, $\mathbb{S}^\infty = \left\{(a_n)_n \in l^2(\mathbb{R})\right\}$ such that $\sum_n a_n^2 = 1$. Prove that the identity and the constant map $(a_1, a_2, \dots) \mapsto (1, 0, \dots)$ are both homotopic to the map $(a_1, a_2, a_3, \dots) \mapsto (0, a_1, a_2, a_3, \dots)$ and deduce that $\mathbb{S}^\infty$ is contractible.

7. Let $\mathbb{RP}^2$ denote the real projective plane.

   (i) What is $\pi_1(\mathbb{RP}^2)$? You may use any valid method but explain your answer .

   (ii) The space $\mathbb{RP}^2$ can be obtained from the unit disk $\mathbb{D}^2$ by identifying $x \sim -x$ if $||x|| = 1$. An element in $\mathbb{RP}^2$ is denoted by $[y]$ where $y \in \mathbb{D}^2$. Let $p \in$int$(\mathbb{D}^2)$, the interior of $\mathbb{D}^2$. Find the fundamental group of $\mathbb{RP}^2 - \{[p]\}$.

8. (i) Let $\pi : \tilde{X} \mapsto X$ be a covering map and q and p the base-points in $\tilde{X}$ and X, respectively, so that $\pi(q) = p$. Put $G = \pi_1(X, p)$ and $H = \pi_1(\tilde{X}, q)$. It is known that a loop α based at p lifts to a loop $\tilde{\alpha}$ in $\tilde{X}$ based at q if and only the equivalence class $\langle\alpha\rangle \in \pi_*(H)$. Prove that the cardinality of $\pi^{-1}(p)$ is the same as in the index of $\pi_*(H)$ in G.

   (ii) Let $X = \mathbb{S}^2 \cup J$ where $J = \{(0, 0, z) \in \mathbb{R}^3 : -1 \leq z \leq 1\}$ is the interval on the z-axis joining the north and south poles. Compute the fundamental group of X.

9. (i) Give the definition of a covering space.
   (ii) Consider the map $p : (0, 3) \to \mathbb{S}^1$ by $p(x) = e^{2i\pi x}$. Is p a covering map? Explain.