Real Analysis, Ph.D. Qualifying Exam

Trieu Le and Sönmez Şahutoğlu

August 15, 2016

Instructions: Do <u>six</u> of the eight questions. You must show all your work and state all the theorems you use. No materials are allowed. 3 hours. **On this exam m stands for Lebesgue measure on $\mathbb{R}$.**

1. Let $\{f_n\}$ be a sequence of real-valued measurable functions on $[0,1]$ and

$$A = \{x \in [0,1] : \lim_{n\to\infty} f_n(x) \text{ exists}\}.$$

   Show that A is measurable.

2. Let $0 < \varepsilon < 1$. Show that there exists a dense open set $E \subset [0,1]$ such that $m(E) \leq \varepsilon$. Can you find $\widetilde{E}$ so that $m(\widetilde{E}) = \varepsilon$?

3. Let $S = \{f \in C^1([0,1]) : f(0) = 0, \|f'\|_\infty \leq 1\}$.
   (a) Show that S has a compact closure in $C([0,1])$ (note that the norm on $C([0,1])$ is the sup-norm).
   (b) Show that S has a compact closure in $L^p([0,1])$ for all $1 \leq p \leq \infty$.

4. Let $\mathcal{A} = \left\{ \sum_{n=1}^{N} c_n(1-x)^n : c_1, \ldots, c_N \in \mathbb{R} \text{ and } N = 1, 2, \ldots \right\}$ and $\phi \in C([0,1])$ with $\phi(1) = 0$. Show that there exists a sequence $\{f_j\} \subset \mathcal{A}$ such that $f_j \to \phi$ uniformly on $[0,1]$ as $j \to \infty$.

5. Let f be a non-negative measurable function on $[0,1]$. Find the limit

$$\lim_{n\to\infty} \int_0^1 \frac{1+n}{1+nf(x)} dm(x).$$

   You must show all details.

6. Let $\{f_n\}$ be a sequence of real-valued continuous functions on $[0,1]$ such that for every $x \in [0,1]$, we have

$$f(x) = \sup\{|f_n(x)| : n = 1, 2, \ldots\} < \infty.$$

   (a) Show that there exists a non-empty open interval $(a,b) \subset (0,1)$ such that

$$\sup\{f(x) : a < x < b\} < \infty.$$

   (b) Find an example of the sequence $\{f_n\}$ such that

$$\sup\{f(x) : 0 < x < 1\} = \infty.$$

7. Let $\{f_n\}$ be a sequence of real-valued measurable functions on $[1, \infty)$ and define $g_n(x) = x f_n(x)$ for all n. Assume that $\{g_n\}$ is a convergent sequence in $L^p([1, \infty))$ for some $1 \leq p \leq \infty$.

   (a) Show that $\{f_n\}$ is convergent in $L^p([1, \infty))$.

   (b) Show that $\{f_n\}$ is convergent in $L^1([1, \infty))$ if $p \neq \infty$.

   (c) Find an example of the sequence $\{f_n\}$ such that $\{g_n\}$ is a convergent sequence in $L^\infty([1, \infty))$ but $\{f_n\}$ does not converge in $L^1([1, \infty))$.

8. Let μ be a finite positive measure on X and $f : X \to [0, \infty)$ be measurable. For each integer $n \geq 1$ denote $A_n = f^{-1}([n, \infty))$. Show that f belongs to $L^1(X, d\mu)$ if and only if $\sum_{n=1}^{\infty} \mu(A_n) < \infty$.