Ph.D. QUALIFYING EXAM
DIFFERENTIAL EQUATIONS
Fall Semester, 2017

Biao Ou, Chunhua Shan

This exam has two parts, ordinary differential equations and partial differential equations. Choose four problems in each part.

Part I: Ordinary Differential Equations

1. Assume $V(x, y) \in C^3(\mathbb{R}^2, \mathbb{R})$. Consider the system

$$\begin{cases} \dot{x} &= V_x(x, y), \\ \dot{y} &= V_y(x, y). \end{cases}$$

Suppose that (x_0, y_0) is an equilibrium of the system, and the determinant of Hessian matrix of $V(x, y)$ at (x_0, y_0) is not equal to zero.

Prove: If (x_0, y_0) is a saddle point of $V(x, y)$, it is a saddle point of the system. If (x_0, y_0) is a local minima or maxima of $V(x, y)$, it is an unstable or a stable node of the system, respectively.

2. Suppose that $u(t)$ and $\alpha(t)$ are two nonnegative, continuous real-valued functions on $[a, b]$, and

$$u(t) \leq C + \int_a^t \big[\alpha(s)u(s) + K\big] ds, \quad a \leq t \leq b,$$

where C and K are nonnegative real numbers.

Prove:

$$u(t) \leq [C + K(t - a)]e^{\int_a^t \alpha(s)ds}, \quad a \leq t \leq b.$$

3. Prove that the planar system

$$\begin{cases} \dot{x} &= x - y - x^3, \\ \dot{y} &= x + y - y^3, \end{cases}$$

has a unique limit cycle in the region $\{(x, y) | 1 < x^2 + y^2 < 2\}$.

Hint: Polar coordinates. You may need the fact that the only equilibrium of system is at the origin.

4. Consider the linear system

$$\dot{x} = B(t)x, \quad x \in \mathbb{R}^n,$$

where $B(t)$ is a continuous, $n \times n$ matrix-valued function for all $t \in \mathbb{R}$.

Let A be an $n \times n$ constant matrix. Suppose that: **(H1)** all eigenvalues of A have negative real part; **(H2)** $\int_0^{+\infty} \|B(t) - A\| dt < +\infty$.

Prove that the solution $x(t) = 0$ is asymptotically stable.

5. Consider the system of linear differential equations $dx/dt = Ax$ where A is a real $n \times n$ matrix with n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $v_1, \ldots, v_n$ be n corresponding eigenvectors. Prove that 1) the set of eigenvectors form a basis for $\mathbb{R}^n$; and 2) any solution of the system is of form below where $c_1, \ldots, c_n$ are constants.

$$x(t) = c_1 e^{\lambda_1 t} v_1 + \cdots + c_n e^{\lambda_n t} v_n.$$

6. Solve the initial value problem and find a sequence of numbers $\{t_n\}$ such that $\lim_{n\to\infty} u(t_n) = \infty$.

$$u'' + 4u = 0.04 \cos 2t - 0.03 \sin 2t; \quad u(0) = 0, \ u'(0) = 0.$$

7. Find all the eigenvalues and eigenfunctions for the Sturm-Liouville problem.

$$u''(t) + \lambda u(t) = 0, \quad u'(0) = 0, \ u(\pi) = 0.$$

Part II: Partial Differential Equations

1. Find a solution to the heat equation with the initial values:

$$\begin{cases} u_t = 4u_{xx}, & \text{for } -\infty < x < \infty,\ t > 0, \\ u(x,0) = 3e^{-2x^2}, & \text{for } -\infty < x < \infty. \end{cases}$$

2. Let u be a positive harmonic function on the ball

$$B_r(0) = \{x \in \mathbb{R}^3 : |x| = \left(\sum_{i=1}^{3} x_i^2\right)^{\frac{1}{2}} \leq r\}.$$

(a) Prove that for any $|x| < r$,

$$\frac{r(r - |x|)}{(r + |x|)^2} u(0) \leq u(x) \leq \frac{r(r + |x|)}{(r - |x|)^2} u(0).$$

(b) Show that if u is a positive harmonic function on $\mathbb{R}^3$, then u is a constant function.

3. Let $\Omega \subset \mathbb{R}^n$ be a connected, bounded open set with a smooth boundary $\partial\Omega$. If $u(x,t)$ is a C^2 function on $\overline{\Omega} \times [0,\infty)$, and solves the initial/boundary-value problem

$$\begin{cases} u_{tt} - 9\Delta u = 0, & (x,t) \in \overline{\Omega} \times [0,\infty) \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty) \\ u(x,0) = 0, u_t(x,0) = 0, & x \in \overline{\Omega} \end{cases} .$$

Prove u is identically equal to zero on $\overline{\Omega} \times [0,\infty)$.

4. Find all the radially symmetric solutions of $\Delta u + u = 0$ on $\mathbb{R}^3$.

5. Solve the initial-boundary value problem for the wave equation.

$$\begin{cases} u_{tt}(t,x) - 4u_{xx}(t,x) = 0 & \text{for } t > 0, x > 0; \\ u(0,x) = xe^{-x} & \text{for } x > 0; \\ u_t(0,x) = \sin x & \text{for } x > 0; \\ u(t,0) = 0 & \text{for } t > 0. \end{cases}$$

6. Find the function $u(x)$ on $\mathbb{R}^2$ satisfying

$$\begin{cases} x_1 u_{x_1} + x_2 u_{x_2} = x_1^2 + x_2^2 & \text{for all } x_1, x_2 \\ u(\cos\theta, \sin\theta) = \cos\theta + \sin\theta & \text{for all } \theta. \end{cases}$$

7. Let $u(x_1, \ldots, x_n)$ be a differentiable function on the closed unit cube Q defined by $0 \leq x_1 \leq 1, \ldots, 0 \leq x_n \leq 1$ and $u = 0$ on the boundary of the cube. Prove that there is a positive constant $C(n)$ such that the inequality holds for all such a function u.

$$\int_Q u^2 dx \leq C(n) \int_Q (u_{x_1}^2 + \cdots + u_{x_n}^2) dx.$$