Department of Mathematics
The University of Toledo

Ph.D. Qualifying Examination

Probability and Statistical Theory

September 23, 2017

Instructions

Do all four problems.

Show all of your computations.
Prove all of your assertions or quote appropriate theorems.
This is a closed book examination.
This is a three hour test.

1. Let $\{X_n, n \geq 1\}$ be a sequence of random variables Show $X_n \xrightarrow{P} 0$ $(r > 0)$ if and only if

$$\mathrm{E}\left(\frac{|X_n|^r}{1+|X_n|^r}\right) \to 0.$$

2. $X_1, \cdots, X_n \overset{iid}{\sim} \mathsf{Bernoulli}(p)$.

   a. Show that $\sqrt{n}(\bar{X}_n - p) \xrightarrow{d} N(0, p(1-p))$.

   b. Find the uniformly minimum variance unbiased estimator (UMVUE) T_n for $g(p) = p(1-p)$.

   c. Show that for $p \neq 1/2$,

$$\sqrt{n}\left\{T_n - p(1-p)\right\} \xrightarrow{d} N(0, (1-2p)^2 p(1-p)).$$

   d. Show that for $p = 1/2$,

$$n\left(T_n - 1/4\right) \xrightarrow{d} -1/4\chi_1^2.$$

3. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $A_n \in \mathcal{F}$ for $n \geq 1$. Show that if there is an $A \in \mathcal{F}$ such that $\sum_{n=1}^{\infty} P(A \cap A_n) < \infty$, then $P(\limsup_n A_n) \leq 1 - P(A)$.

4. Let $X_1, \ldots, X_n$ be a random sample drawn from a population with density $p(x;\theta)$, where θ is a p-dimensional parameter contained in a parameter space Θ. Write $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{x} = (x_1, \ldots, x_n)$, where $\mathbf{x}$ is the observed value of $\mathbf{X}$. Let $\mathcal{X}^n$ denote the sample space of $\mathbf{X}$ and $L(\theta;\mathbf{x}) = \prod_{i=1}^{n} p(x_i;\theta)$ represent the likelihood function of θ based on the data $\mathbf{x}$. For a statistical model $\mathcal{P} = \{p(x;\theta) : \theta \in \Theta \subset \mathcal{R}^p\}$, show that a statistic $T(\mathbf{X})$ is sufficient for θ if for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}^n$, the equality $T(\mathbf{x}) = T(\mathbf{y})$ implies that $L(\theta;\mathbf{y}) = m(\mathbf{x},\mathbf{y})L(\theta;\mathbf{x})$ for all $\theta \in \Theta$, where $m(\mathbf{x},\mathbf{y})$ is some function of $(\mathbf{x},\mathbf{y})$ independent of θ.