Real Analysis, Ph.D. Qualifying Exam

Željko Čučković and Trieu Le

September 16, 2017

Instructions: Do six of the eight questions. You must show all your work and state all the theorems you use. No materials are allowed. 3 hours. In this exam, Lebesgue measure on $\mathbb{R}$ or on any interval is denoted by m.

1. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued Lebesgue measurable functions on $\mathbb{R}$. Show that the following sets are measurable.

   (a) $A = \Big\{x \in \mathbb{R} : \text{ the sequence } \{f_n(x)\}_{n=1}^{\infty} \text{ is strictly increasing}\Big\}$.

   (b) $B = \Big\{x \in \mathbb{R} : \text{ the sequence } \{f_n(x)\}_{n=1}^{\infty} \text{ is unbounded}\Big\}$.

2. Let $f_n(x) = nx^{n-1} - (n+1)x^n$, $x \in (0,1)$. Show that

$$\int_{(0,1)} \sum_{n=1}^{\infty} f_n \, dm \neq \sum_{n=1}^{\infty} \int_{(0,1)} f_n \, dm$$

   and $\sum_{n=1}^{\infty} \int_{(0,1)} |f_n| \, dm = \infty$.

3. Let $A \subset \mathbb{R}$ be a set of finite Lebesgue measure. Suppose $f : A \to [0, \infty)$ is integrable on A. For $\epsilon > 0$, define

$$S(\epsilon) = \sum_{k=0}^{\infty} k\epsilon \, m(A_k), \quad \text{where} \quad A_k = \{x \in A : k\epsilon \leq f(x) < (k+1)\epsilon\}.$$

   Prove that

$$\lim_{\epsilon \to 0} S(\epsilon) = \int_A f \, dm.$$

4. Let S be the set of all functions f that are continuous on $[0,1]$ and differentiable on $(0,1)$ with $\int_0^1 |f'|^2 \, dm \leq 1$. Show that S has a compact closure in $C([0,1])$ with sup-norm $\|\cdot\|_\infty$, where

$$\|g\|_\infty = \sup_{0 \leq x \leq 1} |g(x)|.$$

5. Let f be continuous on the interval $[0,1]$. Show that

$$\lim_{n\to\infty} n \int_{(0,1)} x^{2n} f(x)\, dm(x) = \frac{1}{2} f(1).$$

Suggestion: consider first f a polynomial.

6. For a closed, bounded interval $[a,b]$, let $\{f_n\}$ be a sequence in $C([a,b])$. If $\{f_n\}$ is equicontinuous, does $\{f_n\}$ necessarily have a uniformly convergent subsequence? If $\{f_n\}$ is uniformly bounded, does $\{f_n\}$ necessarily have a uniformly convergent subsequence?

7. Let f belong to $L^p((0,\infty))$ for some $1 \leq p < \infty$. Show that

$$\lim_{x\to\infty} \frac{1}{x^2} \int_{(0,x)} t f(t)\, dm(t) = 0. \qquad (*)$$

Does (*) hold if f belongs to $L^\infty((0,\infty))$? Explain.

8. Construct a sequence $\{f_n\}_{n=1}^\infty \subset L^1([0,1])$ such that $\|f_n\|_1 \longrightarrow 0$ as $n \to \infty$ but for any $t \in [0,1]$, the sequence $\{f_n(t)\}_{n=1}^\infty$ does not converge.