Ph.D. QUALIFYING EXAM
DIFFERENTIAL EQUATIONS
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This exam has two parts, ordinary differential equations and partial differential equations. Choose four problems in each part.

Part I: Ordinary Differential Equations

1. Suppose that $u(t), \alpha(t), \beta(t)$ are real-valued and continuous on $[a, b]$, $\beta(t) \geq 0$ on $[a, b]$, and

$$u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)ds, \quad a \leq t \leq b.$$

Prove:

$$u(t) \leq \alpha(t) + \int_a^t \alpha(s)\beta(s)e^{\int_s^t \beta(\tau)d\tau}ds, \quad a \leq t \leq b.$$

2. Consider the sequence of functions $\{y_k(t)\}_{k=0}^{\infty}$ defined by

$$\begin{cases} y_0(t) &= 0, \\ y_{k+1}(t) &= 2 + \displaystyle\int_0^t \cos^2(\tau)y_k(\tau)d\tau, \quad k = 0, 1, \ldots \end{cases}$$

Prove that $\{y_k(t)\}_{k=0}^{\infty}$ converges uniformly on the finite interval $[-N, N]$ where $N > 0$.

3. Consider the following system in the plane minus the origin written in polar coordinates (r, θ):

$$\dot{r} = 1 - r(1 + \cos(\theta)), \quad \dot{\theta} = 1.$$

(a) Show that there exists an annulus $D = \{(r, \theta) \in \mathbb{R}^2 | 0 < r_1 < r < r_2\}$ which is forward invariant for the system, determining suitable constants r_1 and r_2. This means that if a solution starts at time t_0 in D, it will remain in D for all $t \geq t_0$.

(b) Conclude that there is a periodic orbit inside D invoking a suitable result from the theory of ODEs.

4. Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of

$$y'' + p(x)y' + q(x)y = 0,$$

where $p(x)$ and $q(x)$ are continuous on $\mathbb{R}$. Prove that all the roots of $y_1(x)$ and $y_2(x)$ are isolated, and $y_1(x)$ has exactly one root between any two successive roots of $y_2(x)$.

5. Consider the initial value problem

$$\frac{dx}{dt} = \begin{cases} -x\ln|x|, & x \neq 0 \\ 0, & x = 0 \end{cases}, \qquad x(0) = x_0.$$

where $x \in \mathbb{R}$.

(a) Explain if there is a solution with $x_0 = 0$.

(b) If a solution with $x_0 = 0$ exists, is it unique? Prove it or provide a counterexample.

6. Solve the nonhomogeneous linear system for $x \in \mathbb{R}^2$ with the initial condition.

$$\dot{x} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix} \quad x(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

7. Consider the linear system of differential equations $dx/dt = Ax$. Suppose A is an $n \times n$ upper-triangular matrix with all its diagonal elements equal to one. Show that any solution equals e^t multiplied by a polynomial of t of order less than n.

Part II: Partial Differential Equations

1. Find a solution to the heat equation with the initial values:

$$\begin{cases} u_t - 4u_{xx} = 0, & \text{for } \ -\infty < x < \infty, \ t > 0; \\ u(x,0) = 3e^{-x^2}, & \text{for } \ -\infty < x < \infty. \end{cases}$$

2. Let u be a positive harmonic function on the ball

$$B_r(0) = \{x \in \mathbb{R}^3 : |x| = \left(\sum_{i=1}^{3} x_i^2 \right)^{\frac{1}{2}} \leq r\}.$$

(a) Prove that for any $|x| < r$,

$$\frac{r(r-|x|)}{(r+|x|)^2} u(0) \leq u(x) \leq \frac{r(r+|x|)}{(r-|x|)^2} u(0).$$

(b) Show that if u is a positive harmonic function in $\mathbb{R}^3$, then u is a constant function.

3. Solve the following problem

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, & x > 0, t > 0, \\ u(x,0) = g(x), u_t(x,0) = h(x), & x \geq 0, \\ u(0,t) = 0, & t > 0, \end{cases}$$

where $a > 0$.

4. Let Ω be an open bounded domain in $\mathbb{R}^n$, and T be a positive number. Let $\Omega_T = \Omega \times (0, T]$. The parabolic boundary of Ω_T is defined as $\Gamma_T := \overline{\Omega}_T \backslash \Omega_T$. Assume that $u(x,t) \in C^{2,1}(\Omega_T) \cap C^0(\overline{\Omega}_T)$ satisfies

$$u_t - \Delta u \leq 0 \ \text{ in } \ \Omega_T.$$

Prove that

$$\max_{\overline{\Omega}_T} u = \max_{\Gamma_T} u.$$

5. Find all radially symmetric solutions of $\Delta u = n$ in $\mathbb{R}^n$ for $n \geq 3$.

6. Let $\Omega = \mathbb{R}^n \times (0, \infty)$, and $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Suppose that u solves $u_{tt} - a^2 \Delta u = 0$ in Ω where $a > 0$. Fix $x_0 \in \mathbb{R}^n$, $t_0 > 0$ and consider the cone $K = \{(x,t) : |x - x_0| \leq a(t_0 - t), 0 \leq t \leq t_0\}$. Prove that $u \equiv 0$ within K, if $u \equiv u_t \equiv 0$ on $\{(x,t) : |x - x_0| \leq at_0, t = 0\}$.

7. Consider the initial-boundary value problem

$$\begin{cases} u_t(x,t) + u_x(x,t) = x, & x > 0, t > 0, \\ u(x,0) = \sin x, & x > 0, \\ u(0,t) = t, & t \geq 0. \end{cases}$$

Solve this problem using the method of characteristics. Is the solution continuous along the line $x = t$? What happens instead if we choose as initial value $u(x,0) = \cos x$?