Department of Mathematics
The University of Toledo

Ph.D. Qualifying Examination
Probability and Statistical Theory

September 29, 2018

Instructions

Do all four problems.

Show all of your computations.
Prove all of your assertions or quote appropriate theorems.
This is a closed book examination.
This is a three hour test.

1. Let $X \geq 0$ be a random variable on $(\Omega, \mathcal{A}, P)$ and $\int_\Omega X dP = a$, $0 < a < \infty$. Show the set function ν defined on $\mathcal{A}$ as follows,

$$\nu(A) = \frac{1}{a} \int_A X dP$$

is a probability measure on $\mathcal{A}$.

2. Let $\{X_n, n \geq 1\}$ be a sequence of random variables. Show $X_n \xrightarrow{P} 0$ if and only if

$$\mathrm{E}\left(\frac{X_n^2}{1 + X_n^2} \right) \to 0$$

3. [25 points] Let $\{X_n : n \geq 1\}$ be a sequence of random variables and let c be a constant. Show that if the sequence $\{X_n : n \geq 1\}$ converges in distribution to c, then the sequence $\{X_n : n \geq 1\}$ converges in probability to c.

4. [25 points] Let $\mathcal{P} = \{P_\theta,\ \theta \in \Theta\}$ be a family of distributions, where θ is a p-dimensional parameter contained in a parameter space Θ. Suppose that the distributions P_θ of $\mathcal{P}$ have probability densities $p_\theta = \frac{dP\theta}{d\mu}$ with respect to a σ-finite measure μ. Let $X_1, \ldots, X_n$ be a random sample drawn from a population with density p_θ. Write $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{x} = (x_1, \ldots, x_n)$, where $\mathbf{x}$ is the observed value of $\mathbf{X}$. Show that a necessary and sufficient condition for a statistic $U(\mathbf{X})$ to be sufficient for $\mathcal{P}$ is that for any fixed θ and θ_0, the ratio $\frac{p_\theta(\mathbf{x})}{p_{\theta_0}(\mathbf{x})}$ is a function only of $U(\mathbf{x})$.