Real Analysis, Ph.D. Qualifying Exam

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Instructions: Do six of the eight questions. You must show all your work and state all the theorems you use. No materials are allowed. 3 hours. In this exam, Lebesgue measure on $\mathbb{R}$ or on any interval is denoted by m.

1. Let f be a non-negative measurable function on $[0,1]$ such that for all integers $n \geq 1$,

$$\int_{[0,1]} f^n \, dm \leq \frac{2^n}{n^2}.$$

   Show that $f(x) < 2$ for a.e. $x \in [0,1]$.

2. Let $f, f_1, f_2, \ldots$ be functions in $L^1([0,1]) \cap L^2([0,1])$. Suppose that $f_n \to f$ in L^1-norm.

   (a) Show that the sequence $\{\arctan(f_n)\}$ converges to $\arctan(f)$ in L^1-norm.

   (b) Prove or provide a counterexample to the statement: the sequence $\{(f_n)^2\}$ converges to f^2 in L^1-norm.

3. Let $A \subseteq \mathbb{R}$ be a set of Lebesgue measure zero. Show that there exists $c \in [0,1]$ such that the set $c + A = \{c + x : x \in A\}$ does not contain any rational number.

4. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions that are continuously differentiable on $[0,1]$. Suppose further that, for all integers $n \geq 1$, we have $f_n(0) = 1$ and

$$|f_n'(x)| \leq \frac{1}{\sqrt[4]{x}} \quad \text{for} \quad 0 < x \leq 1.$$

   Show that a subsequence of $\{f_n\}_{n=1}^{\infty}$ must converge uniformly on $[0,1]$.

5. Let f and g be nonnegative and measurable on the interval $[0,1]$. Suppose that

$$f(x)g(x) \geq 1 \text{ for all } x \in [0,1].$$

   (a) Show that

$$\Big(\int_{[0,1]} f \, dm\Big)\Big(\int_{[0,1]} g \, dm\Big) \geq 1.$$

(b) Show that for all positive numbers p and r, we have

$$\left(\int_{[0,1]} f^p \, dm\right)^{1/p} \left(\int_{[0,1]} g^r \, dm\right)^{1/r} \geq 1.$$

6. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous real-valued functions on $\mathbb{R}$. Suppose that for each $x \in \mathbb{R}$, the sequence $\{f_n(x)\}_{n=1}^{\infty}$ is bounded. Show that there exists a non-empty open interval $I \subset \mathbb{R}$ and an integer $N \geq 1$ such that $|f_n(x)| < n$ for all $x \in I$ and all integers $n \geq N$.

7. Let (X, d) be a compact metric space. Suppose that $f : X \to X$ is an isometry (which means that $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$). Show that f is surjective.

8. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that

$$\int_{[0,1]} x^{2n} f(x) \, dm(x) = 0$$

for all positive integers $n \geq 1$. Show that $f(x) = 0$ for all $x \in [0, 1]$.