PhD Qualifying Examination in Topology

Geoffrey Martin Friedhelm Schwarz

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If you believe there is an error on a question in this exam, report this to the proctor. If the proctor does not satisfactorily resolve your concern, you may modify the question so that in your view it is correctly stated but not in such a way that it becomes trivial.

Answer three questions of the following six questions.

Section 1

1. Let $f_\lambda : X \to Y_\lambda$ be a family of maps indexed by a set Λ and let $e : X \to \prod_{\lambda\in\Lambda} Y_\lambda$ be the evaluation map into the Cartesian product defined for $x \in X$ by $e(x)_\lambda = f_\lambda(x)$. Suppose that $U \subset X$ is a subset and for some $\lambda_0 \in \Lambda$, let $V = Y_{\lambda_0} - f_{\lambda_0}(X - U)$. Show that if $\pi_\mu : \prod_{\lambda\in\Lambda} Y_\lambda \to Y_\mu$ is the projection mapping, then $e(X) \cap {\pi_{\lambda_0}}^{-1}(V) \subset e(U)$.
2. Let $\{X_\alpha\}_{\alpha\in A}$ be a collection of T_1-spaces where A is an uncountable well ordered set. Let $x \in \prod_{\alpha\in A} X_\alpha$ and define $S_\beta = \{y \in \prod_{\alpha\in A} X_\alpha | y_\alpha = x_\alpha, \alpha \geq \beta\}$. Show that if α is a limit ordinal, then $S_\alpha = \overline{\bigcup_{\beta<\alpha} S_\beta}$.
3. Suppose that X is a compact set and let $\{C_\alpha\}_{\alpha\in A}$ be a collection of nonempty closed subsets that is closed under finite intersections. If U is an open set such that $\bigcap_{\alpha\in A} C_\alpha \subset U$, show that there is $\alpha_0 \in A$ such that $C_{\alpha_0} \subset U$.
4. Let $A \subset X$ and suppose the $i : X \to A$ is a retraction. (a) Show that i is a quotient map. (b) Show that if X is Hausdorff, then A is closed.
5. Let A be a connected subset of a connected space X and let C be a component of $X - A$. Show that $X - C$ is connected.
6. Suppose that X is compact and Y is Hausdorff. Show that if $f : X \to Y$ is continuous then (a) f is closed (b) if f is surjective, then f is a quotient map (c) if f is bijective, then f is a homeomorphism.

Answer two questions of the following five questions.

Section 2

1. (a) Let P^2 be the two dimensional projective space, show that there does not exist a continuous map $s : P^2 \to S^2$ that satisfies $\pi \circ s = \mathrm{id}_{P^2}$, where π is the covering projection $\pi : S^2 \to P^2$. (b) If $S^1 \subset D^2$ is the boundary of the unit disk in the plane show that there does not exist a continuous map $r : D^2 \to S^1$ that satisfies $r \circ i = \mathrm{id}_{S^1}$ where $i : S^1 \to D^2$ is the inclusion.
2. Let $F : I \times I \to X$ be a continuous homotopy such that $F(s, 0) = f(s)$ and $F(1, t) = l(t)$. Show that F is homotopic to $G : I \times I \to X$ such that $G(s, 0) = f{*}l(s)$ and $G(1, t) = l(1)$.
3. Show that if $f : P^2 \to S^1$ is continuous then f is homotopic to a constant map.
4. Show that if $p : X \to Y$ is a covering projection and if any loop at $y_0 \in Y$ that lifts to a loop at $x_0 \in p^{-1}(\{y_0\})$ always lifts to a loop in X, then p is a regular cover, that is, $p_\sharp\pi_1(X, x_0)$ is a normal subgroup of $\pi_1(Y, y_0)$.
5. Let D be the unit disk in R^2 and let $k > 1$ be a fixed integer. Define an equivalence relation $\sim$ on D as follows. For $x, y \in \partial D$, $x \sim y$ if $y = e^{\frac{2\pi l}{k} i} x$ where $0 \leq l \leq k$ is an integer, and if $x \in D - \partial D$, then $x \sim x$. Let $M = D/\sim$. Compute $\pi_1(M)$.