Topology Ph.D. Qualifying Exam

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This exam has been checked carefully for errors. If you find what you believe to be an error in a question, report this to the proctor. If the proctor's interpretation still seems unsatisfactory to you, you may alter the question so that in your view it is correctly stated, but not in such a way that it becomes trivial.

Section 1

Do 3 of the following 5 problems.

1. Let X be a topological space and let $\mathcal{F}$ denote a family of subsets of X. Consider the set

$$W = \{\mathcal{F} | \mathcal{F} \text{ has the finite intersection property}\}.$$

   Partially order W by inclusion. Show that

   (a) W has a maximal element.

   (b) If $\mathcal{H} \in W$ is a maximal element and if $V \cap H \neq \emptyset$ for all $H \in \mathcal{H}$, then $V \in H$.

2. Prove that none of the spaces $[0, 1]$, $[0, 1)$, and $(0, 1)$ is homeomorphic to any of the others.

3. If Y is compact and Z is Hausdorff, show that the projection map $\pi : Y \times Z \to Z$ is a closed map.

4. Call a topological space X totally disconnected if the only connected subsets of X are one-point sets.

   (a) Show that any product of totally disconnected sets is totally disconnected (in the product topology).

(b) Consider a countably infinite product of discrete two-point spaces, *i.e.*, $\Pi_{i=1}^{\infty}\{0, 1\}$, again using the product topology. Prove that this product is not discrete. Note, however, that by (a), this space is totally disconnected!

5. A family of sets $\mathcal{F}$ is said to be locally finite if for any point $x \in X$ and any open neighborhood. U of x, there are at most a finite number of elements of $\mathcal{F}$ that intersect U non-trivially.

   (a) Show that if $\mathcal{F}$ is a locally finite family of closed sets then $\bigcup_{F\in\mathcal{F}} F$ is closed.

   (b) Show that if $\mathcal{F}$ is a locally finite closed cover of X and $f : X \to Y$ has the property that for $F \in \mathcal{F}$, $f|_F$ is continuous, then f is continuous.

Section 2

Do 3 of the following 5 problems.

1. Let
$$T^2 = \frac{[0,1]\times[0,1]}{\{(x,0)\sim(x,1)\}\cup\{(0,y)\sim(1,y)\}}$$
be the 2-torus and consider the quotient space $X = T^2/\{[(x,0)]\}$. The space X is known as the pinched torus. As a surface it can be represented as sketched below. Find $\pi_1(X)$.

2. Let P^2 be real projective space. The space P^2 can be obtained from the disk D^2 by identifying $x \sim -x$ if $\|x\| = 1$. If $p_1, p_2 \in \operatorname{int} D^2$, show that $P^2 - \{[p_1],[p_2]\}$ is homotopy equivalent to the figure eight, $S^1 \vee S^1$, and find the fundamental group of $P^2 - \{[p_1],[p_2]\}$.

3. Suppose that $F : X \times I \to Y$ is a homotopy such that there is $y_0 \in Y$ $F|X \times \partial I = y_0$. Define $F^{-1}(x,t) = F(x, 1-t)$. Show the $F * F^{-1}$ is homotopic to the constant map y_0. Here $*$ denotes concatenation and is defined when $F|X \times 1 = G|X \times 0$ by the expression
$$F * G(x,t) = \begin{cases} F(x,2t) & 0 \le t \le \frac{1}{2} \\ G(x,2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

4. Let $A \subseteq X$ be a subspace of X and let $i : A \to X$ be the natural inclusion map. We call A a <u>retract</u> of X if there exists a continuous map $r : X \to A$ such that $r \circ i$ is the identity map of A.

   Suppose that $A \subseteq X$ is a retract and that $\pi_1 A$ is normal in $\pi_1 X$. Show that $\pi_1 X \cong \pi_1 A \times (\pi_1 X / \pi_1 A)$.

5. Let $p \colon X \to Y$ be a covering projection and let $\gamma \colon I \to Y$ be a path. Fix a point x_0 in $p^{-1}(\gamma(0))$. Show that there exists a unique path $\tilde{\gamma} \colon I \to X$ with the property that $\tilde{\gamma}(0) = x_0$ and $p \circ \tilde{\gamma} = \gamma$.