Ph.D. Qualifying Exam in Real Analysis
Spring 2000, Time: 3 hours, closed book, no notes.

Answer any six questions to get full credit, providing as much detail as time permits.

1. (a) Let f_n, g_n, f, g be Lebesgue summable, $f_n \to f, g_n \to g$ almost everywhere, $|f_n| \leq g_n$ and $\int g_n \to \int g$. Show that $\int f_n \to \int f$.
   (b) Also show that $\int |f_n| \to \int |f|$ if and only if $\int |f_n - f| \to 0$.

2. Let X be a complete metric space and $X = \cup F_n$, countable union of closed sets F_n. then at least one of F_n has non-empty interior.

3. Assume that f is twice continuously differentiable in the interval (a, ∞) and M_0, M_1, M_2 are the least upper bounds of f, f', f'' respectively. show that
$$M_1^2 \leq M_0 M_2.$$
In what way should the inequality be modified in case of a finite interval.

4. Show that
$$\int_0^\infty \frac{\cos x}{1+x} dx = \int_0^\infty \frac{\sin x}{(1+x)^2}$$
where the left integral converges absolutely and the right integral converges only conditionally. Prove all assertions.

5. Assume f is continuously differentiable in the closed interval $[a, b]$ and $f(a) = 0, f(b) = 0, \int_a^b f(x)^2 dx = 1$. Show that
$$\int_a^b x f(x) f'(x) dx = -\frac{1}{2}$$
and
$$\int_a^b f'(x)^2 dx \int_a^b x^2 f(x)^2 dx > \frac{1}{4}.$$

6. (a) Prove Riemann-Lebesgue lemma: If $f(x)$ is Lebesgue summable on an interval $[a, b]$, then $\int_a^b e^{inx} f(x) dx \to 0$ as $n \to \infty$.
   (b) Let $\{n_k\}$ be an increasing sequence of positive integers and let E be the set of all x for which the sequence $\{\sin n_k x\}$ converges. Show that the measure of E is zero. [Hint: You may assume that f can be approximated by continuously differentiable functions g so that $\int |f - g| dx$ is small.]

7. A mapping $f : \mathbb{R} \to \mathbb{R}$ is open if the image of any open interval is open. Show that any continuous open map is monotone.

8. Suppose that f is Lebesgue integrable on $(-\infty, \infty)$. Show that

$$\lim_{n\to\infty} f(x+n) = 0$$

almost everywhere and the function

$$\int_{-\infty}^{x} f(t)dt$$

is continuous.

9. Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{[\sqrt{n}]}}{n}$$

converges.

10. (a) Gauss' Second Mean Value Theorem is usually stated as follows: Assume f, g are Riemann integrable on $[a, b]$ and g is monotone. Then there exists a ξ in $[a, b]$ such that

$$\int_a^b f(x)g(x)dx = g(a)\int_a^{\xi} f(x)dx + g(b)\int_{\xi}^{b} f(x)dx.$$

Proof in this generality is quite involved. But assuming that g is continuously differentiable, prove Gauss' Theorem.

(b) Further if g is decreasing and non-negative, show that $\int_a^b f(x)g(x)dx = g(a)\int_a^{\xi} f(x)dx$.

11. State Stone-Weierstrass Theorem and using it or otherwise, prove that any continuous even function on $[-\pi/2, \pi/2]$ can be approximated by linear combinations of $\{1, \cos^2 x, \cos^2 2x, \ldots \cos^2 kx, \ldots\}$.