Spring 2001 Ph.D Qualifying Exam in Real Analysis

Time 3 hours, closed book, no notes. Answer any six questions completely to get full credit.

1. (a) State the Stone-Weierstrass theorem.

   (b) If f is a continuous function on $[0,1]$ such that $\int_0^1 x^n f(x)dx = 0$ for $n = 1, 2, 3, \cdots$, then show that $f(x) = 0$ for all $x \in [0,1]$.

   (c) Show that the algebra generated by the set $\{1, x^2\}$ is dense in $C[0,1]$ but fails to be dense in $C[-1,1]$.

2. Show that if $0 < p < q < r \leq \infty$, then $L^q(\mathbb{R}) \subset L^p(\mathbb{R}) + L^r(\mathbb{R})$; that is, each $f \in L^q(\mathbb{R})$ can be written as the sum of a function $g \in L^p(\mathbb{R})$ and a function $h \in L^r(\mathbb{R})$.

3. Show that

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n dx = \lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} dx = 1.$$

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function. Show that

$$\lim_{t\to\infty} \int f(x)\sin(xt)dx = \lim_{t\to\infty} \int f(x)\cos(xt)dx = 0.$$

5. Let $K : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a continuous function with compact support i.e., there exists $c > 0$ such that

$$K(x,y) = 0, \quad \text{if} \quad |x| > c \quad \text{or} \quad |y| > c.$$

   Define $T : L^2(\mathbb{R}^m) \to L^2(\mathbb{R}^n)$ by

$$(Tu)(x) = \int_{\mathbb{R}^m} K(x,y)u(y)dy, \qquad u \in L^2(\mathbb{R}^m), x \in \mathbb{R}^n.$$

   (a) Show that there exists $0 < Q < \infty$, such that

$$\|Tu\|_2 \leq Q\|u\|_2, \qquad u \in L^2(\mathbb{R}^m).$$

   How does Q depend on K?

   (b) Let $\{u_k\}_{k=1}^\infty$ be a sequence in $L^2(\mathbb{R}^m)$ with $\sup_k \|u_k\|_2 = M < \infty$. Show that the sequence $\{Tu_k\}_{k=1}^\infty$ is a uniformly bounded and equicontinuous family of functions that vanish outside B, the closed ball of radius c centered at $0 \in \mathbb{R}^n$.

   (c) Show that there exists a subsequence $\{Tu_{k_j}\}_{j=1}^\infty$ that converges uniformly in B to some $v \in C(B)$.

   (d) Show that $\{Tu_{k_j}\}_{j=1}^\infty$ converges, in $L^2(\mathbb{R}^m)$ norm, to some $w \in L^2(\mathbb{R}^n)$.

6. Let$\mathcal{J}$ be the collection of open intervals, $(a, b), -\infty < a < b < \infty$. Let $m^*((a, b)) = b - a$. Let $m^*(A)$ denote the outer measure of a subset $A \subseteq \mathbb{R}$; that is,

$$m^*(A) = \inf \left\{ \sum_{j=1}^{\infty} M^*(I_j) \mid I_j \in \mathcal{J}, A \subseteq \bigcup_{j=1}^{\infty} I_j \right\}.$$

A subset $E \subseteq \mathbb{R}$ is called measurable iff or all $\epsilon > 0$, there is an open subset $G \subset \mathbb{R}$ such that

$$E \subseteq G, \qquad m^*(G \setminus A) < \epsilon.$$

(a) Show that if $E \subset \mathbb{R}$ and $m^*(E) = 0$, then E is measurable.

(b) Show that there exists disjoint sets $E_1, E_2, \cdots, E_j, \cdots$, such that

$$m^*(\bigcup_{j=1}^{\infty} E_j) \lneqq \sum_{j=1}^{\infty} m^*(E_j).$$

7. (a) Prove or find a counter example to the following statement: *If $f : [0, 1] \to \mathbb{R}$ is continuous and E is a measurable subset of $\mathbb{R}$, then $f(E)$ is also measurable.*

(b) Prove or disprove the previous statement after replacing "continuous" by "uniformly continuous".

8. Suppose $A \subset \mathbb{R}^n, B \subset \mathbb{R}^m$ and $K : B \times A \to \mathbb{R}$ are measurable, and that for some $M_1, M_2 \in (0, \infty)$ we have

$$\int_A |K(x, y)| dy \leq M_1, \text{for } x \in B; \qquad \int_B |K(x, y)| dx \leq M_2, \text{for } y \in A.$$

Fix $p \in (1, \infty)$ and define $T : L^p(A) \to L^p(B)$ by

$$(Tf)(x) = \int_A K(x, y) f(y) dy, \qquad f \in L^p(A), x \in B.$$

(a) Use Hölder's inequality to show that for $f \in L^p(A)$ and $x \in B$ we have

$$|(Tf)(x)|^p \leq M_1^{\frac{p}{q}} \int_A |K(x, y)| \, |f(y)|^p \, dy$$

where $q \in (1, \infty)$ is such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

(b) Use Fubini's theorem and show that for $f \in L^p(A)$

$$\|Tf\|_p \leq M_1^{\frac{1}{q}} M_2^{\frac{1}{p}} \|f\|_p.$$

9. Let (X, ρ) be a metric space and $\alpha \in (0, 1)$. A function $f \in C(X)$ is called Hölder continuous with exponent α if the quantity

$$N_\alpha(f) = \left\{ \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} \right\}$$

is finite. Also define

$$\|f\|_u = \sup_{x \in X} |f(x)|$$

(a) Show that the set

$$\mathcal{F} = \{ f \in C(X) \mid \|f\|_u \leq 1, N_\alpha(f) \leq 1 \}$$

is a closed subset of $C(X)$ in the topology of uniform convergence, i.e. in the norm $\| \cdot \|_u$.

(b) Show that if X is compact, then $\mathcal{F}$ is a compact subset of $C(X)$ in the topology of uniform convergence, i.e. in the norm $\| \cdot \|_u$.

10. Let ϕ denote a convex function in the interval $(0, \infty)$. If $0 < a < b < c < d$, show that $\dfrac{\phi(a) - \phi(b)}{a - b} \leq \dfrac{\phi(c) - \phi(d)}{c - d}$. From this or otherwise show that in any finite interval $[a, b]$ contained in $(0, \infty)$, ϕ is Lipschitz i.e., there exists a constant M such that $|\phi(x) - \phi(y)| \leq M|x - y|$ for all x, y in $[a, b]$.

11. Show that the series

$$\sum_1^\infty \frac{\cos nx}{n^\alpha}$$

converges for all x that are not multiples of 2π and all $\alpha > 0$. What can you say when $\cos nx$ is replaced by x^n?

12. Suppose that f, a function defined on an interval (a, b) satisfies the intermediate value theorem i.e., if f assumes the values y_1, y_2, it assumes all values between y_1, y_2. Show that if f is not continuous, it assumes some value infinitely often.