Spring 2003 Ph.D. Qualifying Exam in Real Analysis

Time 3 hours, closed book, no notes. Answer three questions from each of the parts A & B.

Part A

1. Let E be a subset of $\mathbb{R}^n$ with the property that every continuous function on E is bounded. Prove or disprove: E is compact.

2. (a) Give a careful statement of the Stone-Weierstrass Theorem for the case of continuous complex-valued functions on a compact metric space.

   (b) Let X, Y be compact metric spaces. Show that continuous complex-valued functions of the form

$$F(x,\, y) = \sum_{i=1}^{n} f_i(x) g_i(y),\ x \in X,\ y \in Y \text{ and } n = \{1,\, 2, \cdots\}$$

   where $f_i \in \mathcal{C}(X)$ and $g_i \in \mathcal{C}(Y)$, are dense in the supnorm topology on $\mathcal{C}(X \times Y)$ where

$$||G||_\infty \equiv \sup\{|G(x,\, y)| : (x,\, y) \in X \times Y\},\ G \in \mathcal{C}(X \times Y).$$

3. (a) Let I be a closed interval on the real line with Lebesgue measure, m. For $0 < p < q$ show that the space L^p is contained in L^q and that

$$||f||_p \leq ||f||_q [m(I)]^{\frac{1}{p} - \frac{1}{q}}.$$

   (b) If $f \in L^1(\mathbb{R})$ with respect to Lebesgue measure and $a \in \mathbb{R}$, prove that

$$\int_{-\infty}^{\infty} f(x+a)dx = \int_{-\infty}^{\infty} f(x)dx.$$

4. Define $f(x) = \left[\int_0^x e^{-t^2} dt\right]^2$ and $g(x) = \int_0^1 \frac{e^{-x^2(t^2+1)}}{t^2+1}\, dt.$

   (a) Show that $f'(x) + g'(x) = 0$ and deduce that $f(x) + g(x) = \frac{\pi}{4}$.

   (b) Use (a) to prove that

$$\lim_{x \to +\infty} \int_0^x e^{-t^2} dt = \frac{\sqrt{\pi}}{2}.$$

5. (a) Let $\{f_n\}$ be a sequence of continuous real-valued functions on $[0,\, 1]$ and assume that $f_n \rightrightarrows f$ uniformly on $[0,\, 1]$. Prove or disprove:

$$\lim_{n\to\infty} \int_0^{1-\frac{1}{n}} f_n(x)dx = \int_0^1 f(x)dx.$$

(b) Let $f_n(x) = \frac{1}{n}e^{-n^2x^2}$, $x \in \mathbb{R}$, $(n = 1,\, 2,\, 3\cdots)$.

Show that $f_n \rightrightarrows 0$ uniformly on $\mathbb{R}$, that its derivative $f_n' \to 0$ pointwise on $\mathbb{R}$ but that the convergence of $\{f_n'\}$ is not uniform on any interval containing the origin.

6. (a) What is the formula, in terms of a_n, of the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$? Prove that at any point inside the circle of convergence the power series converges absolutely.

(b) Suppose that $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence, 2. Given that k is a fixed positive integer, find the radii of convergence of the following series.

i. $\displaystyle\sum_{n=0}^{\infty} a_n^k z^n$, ii. $\displaystyle\sum_{n=0}^{\infty} a_n z^{kn}$, iii. $\displaystyle\sum_{n=0}^{\infty} a_n z^{n^2}$

Part B

1. Let $f : \mathbb{R} \to \mathbb{R}$. For each $x \in \mathbb{R}$, define

$$\omega(x) = \inf\{\delta(f(U)) : U \text{ a neighborhood of } x\}$$

where, if $E \subset \mathbb{R}$, $\delta(E) \equiv \sup\{|x - y| : x,\, y \in E\}$.

Prove the following:

(a) The function f is continuous at x if and only if $\omega(x) = 0$.

(b) For each $\alpha \in \mathbb{R}$ the set $\{x \in \mathbb{R} : \omega(x) < \alpha\}$ is open.

(c) The set $\{x \in \mathbb{R} : f(x) \text{ is continuous}\}$ is a G_δ set.

(d) There is no real-valued function, f, on $\mathbb{R}$ such that $\{x \in \mathbb{R} : f(x)$ is continuous $\} = Q$, the rational numbers in $\mathbb{R}$.
(Hint: For (d) use the Baire Category Theorem to show that Q cannot be a G_δ set in $\mathbb{R}$).

2. Let $(X,\, d)$ be a compact metric space. A function $f : X \to \mathbb{R}$ is said to be Lipschitz continuous if

$$||f||_d \equiv \sup\left\{\frac{|f(x) - f(y)|}{d(x,\, y)} : x \in X,\ y \in Y,\ x \neq y\right\} < \infty.$$

Denote by $\text{Lip}(X,\, d)$ the collection of all Lipschitz continuous functions on X.

(a) Prove that $\text{Lip}(X,\, d)$ is a Banach space under the norm

$$||f|| = ||f||_\infty + ||f||_d$$

where $||f||_\infty = \max\{|f(x)| : x \in X\}$. Show that the multiplicative inequality

$$||fg|| \leq ||f|| \cdot ||g||,\ f,\, g \in \text{ Lip}(X,\, d)$$

is also true.

(b) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $\text{Lip}(X,\, d)$ with $||f_n|| \leq 1$. Show that there is a subsequence $\{f_{n_k}\}$ and $f \in \text{Lip}(X,\, d)$ such that $f_{n_k} \Rightarrow f$ uniformly on X.

3. (a) Let f be a Lebesgue integrable function on the real line. Show that given $\varepsilon > 0$ there is a $\delta > 0$ such that whenever A is a Lebesgue measurable subset of $\mathbb{R}$ then

$$m(A) < \delta \Rightarrow \int_A |f| dx < \varepsilon.$$

(b) If $f \in \mathcal{C}[0, 1]$, show that $\lim_{n\to\infty} ||f||_n$ exists and compute this limit where

$$||f||_n = \left[\int_0^1 |f|^n dx\right]^{\frac{1}{n}} \quad (n = 1, 2, \cdots).$$

4. (a) Suppose $f, f_n (n = 1, 2, 3, \cdots)$ are real-valued Lebesgue measurable functions on $\mathbb{R}$. Define what is meant by saying $f_n \to f$ in m-measure. (Here m is Lebesgue measure on $\mathbb{R}$).

(b) If we identify Lebesgue measurable functions on $\mathbb{R}$ that agree almost everywhere $[m]$, show that

$$d(f, g) \equiv \int_{\mathbb{R}} \frac{|f - g|}{1 + |f - g|} \, dm$$

is a metric on the space of Lebesgue measurable functions on $\mathbb{R}$.

(c) Show that $f_n \to f$ in m-measure if and only if $\lim_{n\to\infty} d(f_n, f) = 0$.

5. (a) Give careful statements of the Lebesgue Monotone Convergence Theorem and the Lebesgue Dominated Convergence Theorem.

(b) Use these theorems to establish the following:

i. $$\lim_{n\to\infty} \int_1^n (1 - \tfrac{x}{n})^n \ln x \, dx = \int_1^\infty e^{-x} \ln x \, dx.$$

ii. $$\lim_{n\to\infty} \int_0^1 (1 - \tfrac{x}{n})^n \ln x \, dx = \int_0^1 e^{-x} \ln x \, dx.$$