Ph.D. QUALIFYING EXAM
DIFFERENTIAL EQUATIONS
Spring, 2004

This exam has two parts, ordinary differential equations and partial differential equations. In Part I, do problems 1 and 2 and choose two from the remaining problems. In Part II, Choose three problems.

Part I: Ordinary Differential Equations

1. Consider the differential equation with initial condition

$$dx/dt = F(t, x), \quad x(a) = x_0 \in R^n$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ and
$F(t, x) = (F_1(t, x), F_2(t, x), \ldots, F_n(t, x))^T$. Suppose $F(t, x)$ is continuous for $a \le t \le b$ and $x \in R^n$ and satisfies a Lipschitz condition $|F(t, x) - F(t, y)| \le L|x - y|$ for $a \le t \le b$ and all x, y.

(a) Convert the differential equation with the initial condition into an equivalent integral equation.

(b) Set up the Picard iteration process and prove that the sequence converges uniformly on the interval [a, b] to a limit function $x_\infty(t)$.

(c) Show that $x_\infty(t)$ is a solution to the differential equation on [a, b].

(d) Establish that the solution to the differential equation with the given initial condition is unique.

2. (a) State and prove the Sturm Separation Theorem and Sturm Comparison Theorem for O.D.E's of the form $y'' + p(x)y = 0$.

(b) Consider any non-trivial solution to $y'' + xy = 0$. Show that $y(x)$ has infinitely many zeros on $(0, \infty)$ but at most one zero on $(-\infty, 0)$.

3. For the system show that (1,1) and (-1,-1) are its critical points. Linearize the DE about each of the critical points and use the result to describe the nature of the solutions in a neighborhood of the critical points.

$$\dot{x} = x - y \qquad \dot{y} = 4x^2 + 2y^2 - 6$$

4. Find all the eigenvalues and eigenfunctions of the Sturm-Liouville system. Be sure to consider the possibility of $\lambda \leq 0$.

$$y''(t) + \lambda y(t) = 0; \quad y(0) = 0, \ \ y'(1) = 2y(1).$$

5. Find the solution $u(x, y)$ to the integrable system on the domain $\{(x, y) \mid x < 1, y < 1.\}$ with the initial condition.

$$u_x = \frac{u}{2(x+1)}, \quad u_y = \frac{u}{y+1}; \quad u(0,0) = 1.$$

6. Find the fundamental solution matrix for the linear homogeneous system of differential equations. Sketch the possible trajectories.

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

Part II: Partial Differential Equations

1. Solve the quasilinear partial differential equation by the method of characteristics.

$$u_x + u_y = u^2, \quad u(x, 0) = x^2.$$

2. Consider the partial differential equation $u_y = u_x^3$.

(a) Find generalized solutions $u = G(x, y, a, b)$ which are linear in x and y.

(b) Use the method of envelopes to solve the initial value problem when $u(x, 0) = x^2$.

3. Derive (from scratch) the D'Alembert solution to the one-dimensional wave equation

$$\begin{aligned} &u_{tt} = c^2 u_{xx} \ \text{ for } \ x \in R \ \text{ and } \ t > 0; \\ &u(x, 0) = f(x), \ \ u_t(x, 0) = g(x) \end{aligned}$$

where $f(x), g(x) \in C^2(R)$.

4. Suppose $u(x)$ is a harmonic function on an open domain Ω. Prove that if $u(x)$ has a maximum at an interior point then $u(x)$ is a constant function.

5. Let $u(x)$ be a harmonic function on an entire Euclidean space R^n. Suppose there exist two positive constants c_1, c_0 such that

$$|u(x)| \leq c_1|x| + c_0$$

for all x in R^n. Prove that $u(x)$ is a linear function.