Qualifying exam in algebra, April 2005

Do four of the following problems. Please give complete solutions. Two half solutions count less than one complete solution. Indicate clearly which four problems you wish to have graded.

1. Let $K = \mathbb{F}_4$ and $L = \mathbb{F}_{64}$, where $\mathbb{F}_q$ denotes the field with q elements.

a. Show that K is (isomorphic to) a subfield of L.

b. Find a polynomial over K whose splitting field is L.

c. Determine the Galois group of L/K.

2. Suppose that G is a group.

a. For $g \in G$ let c_g denote the map $x \mapsto x^g = g^{-1}xg$. Show that the map $G \to \mathrm{Aut}\,(G)$ given by $g \mapsto c_g$ is a group homomorphism, and identify its kernel.

b. A group is said to be *complete* if the map in part (a) is an isomorphism. Suppose that N is a normal subgroup of G, and that N is complete. Show that $G \cong N \times G/N$.

3. The ring R is *prime* if whenever A and B are ideals of R with $AB = 0$ then either $A = 0$ or $B = 0$.

a. Show that R is prime if and only if whenever $a, b \in R$ such that $aRb = 0$ then either $a = 0$ or $b = 0$.

b. Give an example of a prime ring that is not a domain. Verify both that it is prime and that it is not a domain.

4. Let R be a commutative domain with identity and let S be a be a subset with the following properties: $0 \notin S$; $1 \in S$; if $a, b \in S$ then $ab \in S$. Let Q denote the field of quotients of R, and set

$$R_S = \{as^{-1} \in Q : a \in R, s \in S\}.$$

You may use without proof that R_S is a subring of Q which contains R.

a. Show that if R is a PID then so is R_S.

b. Suppose $R = \mathbb{Z}$, p is a prime, and

$$S = \{n \in \mathbb{Z} : p \text{ does not divide } n\}.$$

Find all ideals of R_S.

5. A permutation $\alpha \in \mathrm{Sym}_n$ is *regular* if *all* of the cycles in the cycle decomposition of α have the same length. Note that the only regular permutation which fixes a point is the identity.

a. Prove that α is regular if and only if α is a power of an n-cycle.

b. Let G be a finite group and let $a \in G$. Prove that the permutation $\alpha\colon x \mapsto ax$ on G is regular.

6. If $\sigma\colon F \to F_1$ is an isomorphism of fields, let $\overline{\sigma}\colon F[x] \to F_1[x]$ be defined by the rule

$$\overline{\sigma}\left(\sum a_i x^i\right) = \sum \sigma(a_i)x^i.$$

a. Show that $\overline{\sigma}$ is an isomorphism.

b. Suppose that $f(x) \in F(x)$ and let $f_1(x) = \overline{\sigma}(f_1(x))$. Let L be a splitting field of $f(x)$ over F and let L_1 be a splitting field of $f_1(x)$ over $F_1[x]$. Show that there is an isomorphism $\tau\colon L \to L_1$ such that $\tau(a) = \sigma(a)$ for every $a \in F$.

b. Give an example to show that there may be more than one choice for τ.