Ph.D. Qualifying Exam

March 2005

Instructions:

1. If you think that a problem is incorrectly stated ask the proctor. If his explanation is not to your satisfaction, interpret the problem as you see fit, but not so that the answer is trivial.

2. From each part solve 3 of the 5 five problems.

3. If you solve more than three problems from each part, indicate the problems that you wish to have graded.

Part A

1. Suppose that $a_n > 0$ and suppose that there is a sequence $b_k > 0$ and a real number $c > 0$ for which if $j > 1$, $\dfrac{a_j}{a_{j+1}} b_j - b_{j+1} > c$. Show that $\sum_{n=1}^{\infty} a_n$ converges.

2. Suppose that $f(x)$ is continuous on $[a, b]$ and is differentiable on $[a, c) \cup (c, b]$. If $\lim_{x\to c} f'(x) = \ell$ then $f(x)$ is differentiable at c and $f'(c) = \ell$.

3. Suppose that X is a complete metric space and that $f\colon X \to \mathbf{R}$ is a nowhere continuous function. Show that there is a open set $J \subset X$ and a real number $\alpha > 0$ with the property that for any ball $B \subset J$, $\operatorname{diam}(f(B)) > \alpha$.

4. Show that if $f(x)$ is of bounded variation on $[a, b]$, then for any $t \in [a, b]$, $\lim_{x\to t^+} f(x)$ and $\lim_{x\to t^-} f(x)$ exist. If you use the fact that $f(x)$ is a difference of two monotone functions, then you must prove that also.

5. Prove that in the uniform norm the space of step functions defined on $[0, 1]$ is not separable, but the space of continuous functions defined on $[0, 1]$ is separable.

Part B

1. Show that $\dfrac{1}{2} \log\left(\dfrac{1+x}{1-x}\right) = \displaystyle\sum_{n=1}^{\infty} \frac{x^{2n+1}}{2n+1}$, and integrate this identity to show that to show that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \log 2$. Carefully justify the steps of your calculation.

2. Show that a simple function on $[0,1]$ can be approximated almost uniformly by a sequence of step functions defined on $[0,1]$.

3. Suppose that $f_n(x)$ is a sequence of positive integrable functions that converge in measure to a measurable function $f(x)$. Show that if for any n $\int f_n(x)dx \le A$, then $\int f(x)dx \le A$.

4. Suppose that $f(x)$ is an integrable function on $\mathbf{R}$ and $a > 0$. Show that $\sum_{n=0}^{\infty} f(\frac{x}{a}+n)$ converges absolutely almost everywhere on $[0,a]$ to an integrable function.

5. Recall that $f(x)$ is essentially bounded if there exists a measurable subset A with $\mu(A^{\sim}) = 0$ and $\sup_A |f(x)| < \infty$, where $A^{\sim}$ denotes the complement of A. Let $\mathcal{A}$ be the set of all such subsets and let $||f|| = \inf_{\mathcal{A}}(\sup_{A \in \mathcal{A}} |f(x)|)$. Show that there is a set $A_0 \in \mathcal{A}$ such that $||f|| = \sup_{A_0} |f(x)|$, and show that if $f(x)$ and $g(x)$ are essentially bounded then so is $f(x)g(x)$ and $||fg|| \le ||f||||g||$.