Ph.D. Qualifying Exam

April 16, Spring 2005

Instructions:

1. If you think that a problem is incorrectly stated ask the proctor. If his explanation is not to your satisfaction, interpret the problem as you see fit, but not so that the answer is trivial.

2. From each part solve 3 of the six problems.

3. If you solve more than three problems from each part, indicate the problems that you wish to have graded.

Part A

1. Show that there does not exist a continuous one to one function taking $[0, 1]$ onto $[0, 1] \times [0, 1]$.

2. Let M be the bounded sequences of complex numbers with the sup norm $||a|| = \sup_n(|a_n|)$. Show that M is a Banach space, and that the subspace of convergent sequences is a closed subspace.

3. If $f(x)$ is differentiable on $[a, b]$ and $f'(a) < c < f'(b)$ show that there is $a < x < b$ such that $f'(x) = c$.

4. Suppose that f is infinitely differentiable on $[a, b]$ and suppose that for any $a \leq x \leq b$ the Taylor series of $f(x)$ has positive radius of convergence at x. Use the Baire Category Theorem to show that $f(x)$ must be analytic on a subinterval of $[a, b]$. Hint: Recall that the radius of convergence $\rho(z)$ is given by the expression $1/\rho(z) = \limsup_n \sqrt[n]{\frac{f^{(n)}(z)}{n!}}$.

5. Given any finite sequence of positive reals $\{x_1, \ldots, x_n\}$, we define the arithmetic mean as $(x_1+x_2+\ldots x_n)/n$ and the geometric mean as $(x_1x_2\ldots x_n)^{1/n}$. Prove that the arithmetic mean is greater than or equal to the geometric mean.[Hint: Treat it under maxima-minima.]

6. Suppose g is a continuous function on $\mathbb{R}$, $g(0) = 0$, and g' is bounded in absolute value by M on $\mathbb{R}$. Then show that the series

$$\sum \frac{1}{n} g(\frac{x}{n})$$

converges and the sum is continuous. Is it differentiable?

Part B

1. If f is real valued measurable on an interval I. Sow that there exists a sequence $\{\phi_n\}$ of continuous functions converging to f in measure.

2. Suppose ϕ is infinitely differentiable in $(-1, 1)$ and vanishes in a neighborhood of -1 and 1. Show that for any natural number N, there exists a constant $C = C_N$ such that

$$\left| \int_{-1}^{1} e^{i\lambda t} \phi(t) dt \right| \leq C\lambda^{-N}$$

for all $\lambda > 0$.

3. Let $X \subset \mathbf{R}$ be a set of finite measure and suppose that $f(x)$ defined on X is measurable. Let $\rho(f) = \int_X \frac{|f|}{1+|f|} dx$. Show that $\rho(f)$ is well defined and that the sequence of function $f_n(x)$ defined on X converges in measure if and only if $\lim_{n\to\infty} \rho(f_n) = 0$.

4. Suppose $\mu(X) < \infty$ and $f_n(x)$ converges to $f(x)$ in measure. Show that ${f_n}^2(x)$ converges to $f^2(x)$ in measure. Hint: write ${f_n}^2 - f^2 = (f_n - f)^2 + 2f(f_n - f)$.[Hint: First prove when f is bounded.]

5. Suppose that $f(x)$ is integrable on $[a, b]$ and for any $a \leq c < d \leq b$, $\int_c^d f(x)dx = 0$ Show the $f(x) = 0$ almost every where.

6. Evaluate the limit of the sequence $\displaystyle\int_0^\infty n \sin(x/n)(x(1 + x^2))^{-1} dx$ of integrals as n tends to ∞. Justify the convergence of the integrals as well as the existence of the limit.