Ph.D. QUALIFYING EXAM
DIFFERENTIAL EQUATIONS
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This exam has two parts, ordinary differential equations and partial differential equations. In Part I, do problems 1 and 2 and choose two from the remaining problems. In Part II, Choose three problems.

Part I: Ordinary Differential Equations

1. Consider the differential equation with initial condition

$$dx/dt = F(t, x), \quad x(a) = x_0 \in R^n$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ and
$F(t, x) = (F_1(t, x), F_2(t, x), \ldots, F_n(t, x))^T$. Suppose $F(t, x)$ is continuous for $a \le t \le b$ and $x \in R^n$ and satisfies a Lipschitz condition $|F(t, x) - F(t, y)| \le L|x - y|$ for $a \le t \le b$ and all x, y.

(a) Convert the differential equation with the initial condition into an equivalent integral equation.

(b) Set up the Picard iteration process and prove that the sequence converges uniformly on the interval [a, b] to a limit function $x_\infty(t)$.

(c) Show that $x_\infty(t)$ is a solution to the differential equation on [a, b].

(d) Establish that the solution to the differential equation with the given initial condition is unique.

2. Consider the second order equation

$$y'' + p(x)y' + q(x)y = 0$$

on an open interval where $p(x), q(x)$ are continuous.

1) Let $y(x)$ be a non-zero solution. Prove that all the roots of $y(x)$ are isolated.

2) Let $y_1(x), y_2(x)$ be two linearly independent solutions. Prove that between any two roots of $y_1(x)$ there is exactly one root of $y_2(x)$.

3. Consider the nonlinear DE $\ddot{x} + x^3 = 0$.

(a) Solve the DE and sketch the trajectories in the phase plane.

(b) Show that every orbit is periodic with a critical point at (0,0).

(c) Express the period of any orbit as a function of its amplitude $P = P(x_0)$. Show that

$$\lim_{x_0 \to 0+} P(x_0) = +\infty, \qquad \lim_{x_0 \to \infty} P(x_0) = 0.$$

4. Consider the Sturm-Liouville system

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) + y'(\pi) = 0.$$

(a) Find all eigenfunctions and eigenvalues $(y_n(x), \lambda_n)$. Be sure to check the possibilities $\lambda_n \leq 0$.

(b) Show that the set of eigenfunctions form an orthogonal set of functions in $L^2[0, \pi]$.

(c) Solve for the λ_n graphically and make a good asymptotic estimate for λ_n.

5. (a) Find the fundamental solution matrix for the linear system

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

(b) Solve the initial value problem.

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^t, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

6. Suppose a non-negative function $\sigma(t)$ satisfies $\sigma(0) = 0$ and

$$-\sigma(t) \leq \sigma'(t) \leq \sigma(t).$$

Prove that $\sigma(t) \equiv 0$.

Part II: Partial Differential Equations

1. (Poisson's Formula on a Disk) Let $f(\theta)$ be a continuous and 2π-periodic function with Fourier series

$$f(\theta) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos k\theta + b_k \sin k\theta).$$

Let

$$u(r,\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} r^k(a_k \cos k\theta + b_k \sin k\theta).$$

(a) Prove that the series for $u(r,\theta)$ converges uniformly on any disk $B_R = \{(r,\theta) \mid 0 \le r \le R\}$ with $R < 1$.

(b) Show how to rewrite the series for $u(r,\theta)$ in the form

$$u(r,\theta) = \int_0^{2\pi} f(\phi)P(r,\theta-\phi)d\phi$$

where P is the Poisson kernel satisfying

$$P(r,\phi) = \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos\phi + r^2}.$$

(c) Prove that $\lim_{r\to 1^-} u(r,\theta) = f(\theta)$ uniformly.

2. Consider the Fourier series solution to the heat equation with an initial condition and a boundary condition

$$\begin{cases} u_t = u_{xx}, & \text{for } 0 < x < \pi,\ t > 0, \\ u(x,0) = f(x), & \\ u(0,t) = u(\pi,t) = 0. & \end{cases}$$

(a) Derive the formal solution $u(x,t) = \sum_{k=1}^{\infty} b_k e^{-k^2 t} \sin kx$ where the b_k are the coefficients of the Fourier series $\sum_{k=1}^{\infty} b_k \sin kx$ for the continuous function $f(x)$.

(b) Show that for every $\delta > 0$ this solution series converges uniformly in the region $0 \le x \le \pi, t \ge \delta$. Also show the same convergence for the corresponding series for u_t, u_x, u_{xx}.

(c) Show that $\lim_{t\to 0^+} u(x,t) = f(x)$ in the L^2 norm.

3. Find a solution to the initial value problem

$$\begin{aligned} u_t(x,t) &= 9u_{xx}(x,t) \quad \text{on} \quad t>0 \\ u(x,0) &= 4\exp(-2x^2). \end{aligned}$$

4. Prove the mean value property of harmonic functions and use it to prove the strong maximum principle for harmonic functions

5. Let $B^+ = \{(x,y) \mid x^2+y^2<1, y>0\}$ be the open half disk. Suppose $u(x,y) \in C^2(B^+) \cap C^0(\bar{B^+})$ satisfies $\Delta u = u_{xx}+u_{yy} = 0$ in B^+ and $u(x,0) = 0$. Prove that the extension $u(x,y)$ to B defined as follows is a harmonic function on all B.

$$u(x,y) = \begin{cases} u(x,y) & \text{if } y \geq 0; \\ -u(x,-y) & \text{if } y<0. \end{cases}$$