Department of Mathematics
The University of Toledo

Ph.D. Qualifying Examination
Probability and Statistical Theory

April 15, 2006

Instructions

Do all four problems.

Show all of your computations.
Prove all of your assertions or quote appropriate theorems.
This is a closed book examination.
This is a three hour test.

1. Let X and Y be independent random variables such that X and $X - Y$ are independent. Prove that X must be degenerate, i.e., $P(X = c) = 1$ for some constant c.

2. Let F be a cumulative distribution function on the real line $\mathbf{R}$ and $a \in \mathbf{R}$. Show that

$$\int [F(x + a) - F(x)]dx = a.$$

3. Let $X_1, \cdots, X_n$ be an i.i.d. sample with density $\frac{1}{\theta}\exp(-x/\theta)I[x > 0]$, where $\theta > 0$ is the parameter. Consider the action space $\mathcal{A}$=$(0, \infty)$ and the loss function $L(\theta, a) = \frac{(\theta-a)^2}{\theta^2}$.

a). Show that $\delta(X) = \frac{\sum_{i=1}^n X_i}{n+1}$ is an equalizer rule.

b). Show that $\delta(X)$ is a minimax rule.

4. Let $X_1, \cdots, X_n$ be i.i.d. random variables with the density
$f(x;\theta) = \frac{2}{\pi\theta}\exp(-\frac{x^2}{\pi\theta^2})I[x \geq 0]$.
where $\theta > 0$ is an unknown parameter. Note that
$E(X_1) = \theta, E(X_1^2) = \frac{\pi}{2}\theta, E(X_1^4) = \frac{3\pi^2}{4}\theta^4$.

a). Find the Cramer-Rao lower bound for the variance of an unbiased estimator of θ^2. Is there a UMVUE of θ^2 whose variance is the same as the C-R lower bound? If so, what is the UMVUE?

Note that the random variable $Y = \frac{2}{\pi\theta^2}\sum_{i=1}^n X_i^2$ has a χ_n^2 distribution with pdf
$f(y) = \frac{1}{2^{n/2}\Gamma(n/2)}e^{-y/2}y^{n/2-1}I[y > 0]$.

b). Use the above results to find a multiple of $(\sum X_i^2)^{1/2}$ that is an unbiased estimator of θ. Is this estimator a UMVUE of θ? Is the sample mean, $\bar{X}$, a UMVUE of θ? Explain your answers.

c). Find the most powerful size α test for $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$. Here, θ_0 and $\theta_1 > \theta_0$ are given positive numbers.